

Exterior Differential Systems, from Elementary to Advanced

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ABSTRACT. This monograph was developed to support a series of lectures at the Institute of Mathematics at the Polish Academy of Sciences in September 2016, as part of a Workshop on the Geometry of Lagrangian Grassmannians and Nonlinear PDEs.

The goal is to cut the shortest-possible expository path from the common, elementary concepts of geometry (linear algebra, vector bundles, and algebraic ideals) to the advanced theorems about the characteristic variety. Hopefully, these lectures lower the barrier to advanced topics in exterior differential systems by exposing the audience to elementary versions of several key results regarding the characteristic variety, and to outline how these results could be used to push the frontiers of the subject. These key results are:

- (i) The incidence correspondence of the characteristic variety,
- (ii) Guillemin normal form and Quillen's thesis,
- (iii) The Integrability of Characteristics (Guillemin, Quillen, Sternberg, Gabber),
- (iv) Yang's Hyperbolicity Criterion.

To accomplish this, the subject of exterior differential systems is reinterpreted as the study of smooth sub-bundles of the Grassmann bundle over a smooth manifold. These notes intentionally obscure the role of exterior differential forms in computation, instead focusing on tableaux as subspaces of homomorphisms and on symbols as varieties of endomorphisms. Techniques involving differential ideals or PDE analysis are hardly mentioned. Instead, Guillemin normal form (a generalization of Jordan decomposition from a single endomorphism to a variety of endomorphisms) is the primary computational tool.

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0. Introduction

Given a system of partial differential equations [PDEs] over a manifold, does the system of PDEs have any local solutions to the Cauchy problem? That is, given initial conditions on a locally-defined hypersurface, can we produce a local solution that satisfies those initial conditions and also satisfies the PDEs? More generally, which initial hypersurfaces admit such solutions? Can we do this repeatedly, by solving a sequence of initial-value problems from dimension 0 to 1, 1 to 2, and so on to build solutions successively at any point? What is the shape of the family of local solutions obtained in this way? More generally still, how can we determine whether two systems of PDEs are “the same” up to local coordinate transformations? Does the space of all PDEs (up to local coordinate transformation) have any meaningful shape or structure of its own?

These are the questions at the heart of the subject of exterior differential systems [EDS]. However—despite the name—differential forms are not themselves the core idea in the study of exterior differential systems. Rather, the core idea is to recognize that these questions are more geometric than analytic, and that ideals (that is, conditions defined by functions) and varieties (that is, shapes cut out by functions) must come into play. To describe families of solutions, we need the geometric language of bundles, schemes, and moduli.

Indeed, it is common for researchers entering the subject to become trapped in an endless cycle of translating systems from local jet coordinates to differential forms and back again, without gaining any new geometric insights and without using the most powerful theoretical ideas in EDS. In particular, it can take many years for researchers to appreciate the central role that the characteristic variety plays in uncovering geometric insights.

The goal of these notes is to cut the shortest-possible expository path from the common, elementary concepts of geometry (linear algebra, vector bundles, and algebraic ideals) to the advanced theorems about the characteristic variety. Hopefully, these lectures lower the barrier to advanced topics in exterior differential systems by exposing the audience to elementary versions of several key results regarding the characteristic variety, and to outline how these results could be used to push the frontiers of the subject. These key results are:

- (i) The incidence correspondence of the characteristic variety,
- (ii) Guillemin normal form and Quillen’s thesis,
- (iii) The Integrability of Characteristics (Guillemin, Quillen, Sternberg, Gabber),
- (iv) Yang’s Hyperbolicity Criterion.

The required background for these lectures is graduate-level linear algebra (short-exact sequences, dual spaces, the rank-nullity theorem, tensor products, generalized eigenspaces, as in Artin’s *Algebra* [Art91]), the fundamentals of smooth manifolds (tangent spaces, Sard’s theorem, bundles, as in Milnor’s *Topology from the Differential Viewpoint* [Mil97]) and basic algebraic geometry (projective space, ideal, variety, scheme, as in Harris’ *Algebraic Geometry, a first course* [Har92]). These lectures assume that the audience has a general cultural awareness of PDEs or EDS in some form but the required definitions are provided.

Even so, it is wise always to have Bryant, et al.'s *Exterior Differential Systems* [BCG⁺90] and Ivey and Landsberg's *Cartan for Beginners* [IL03] nearby. They are cited for comparison frequently.

Everything in these pages can be found in numerous places in the literature in some form or other; I have indicated my favorite sources throughout. The only innovations here are in presentation:

- (i) The central topic is the C^∞ characteristic variety, not the C^ω Cartan–Kähler theorem. This is because I am interested in the question “what does the family of all PDEs look like?” not “how do I solve this particular PDE?”
- (ii) Guillemin normal form for tableaux and symbols plays the central computational role, not differential forms. This is because most humans—and their computer algebra systems—are more comfortable with matrices than with exterior algebra.
- (iii) Exterior differential ideals are not introduced until absolutely needed. This is because many of the essential lemmas depend only on the geometry of the Grassmann variety, which is the variety of the trivial exterior differential system.

For readability, many proofs are omitted or reduced to discussion in prose. This should not be an impediment to understanding or to precision. Most of the proofs are basic linear algebra (in fact, almost all the key results are restatements of the rank-nullity theorem), and details are provided in the references.

Part I

Matrices and Subspaces

1. Tableaux and Symbols

Tableaux are very simple objects; every undergraduate encounters the example “ $r \times n$ matrices form a vector space using the usual matrix operations,” and a tableau is any subspace of that vector space.

Given vector spaces or projective spaces W and V , a *tableau* is a linear subspace of $A \subset \text{Hom}(V, W)$. We use the notation $W \otimes V^*$ and $\text{Hom}(V, W)$ interchangeably, and eventually we switch from vector spaces to complex projective spaces for algebraic convenience.

Being a subspace, any tableau A is the kernel of some linear map σ , called the *symbol*, whose range is written as $H^1(A)$. We have a short exact sequence of spaces:

$$(1.1) \quad 0 \rightarrow A \rightarrow W \otimes V^* \xrightarrow{\sigma} H^1(A) \rightarrow 0,$$

where $H^1(A)$ is just notation for $(W \otimes V^*)/A$. Let $\dim A = s$ and $\dim H^1(A) = t = nr - s$.

For example, let $W = \mathbb{R}^3$ and $V = \mathbb{R}^3$, and consider the 5-dimensional tableau A described in the standard bases by

$$(1.2) \quad \left\{ \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} : \alpha_i \in \mathbb{R} \right\}.$$

The symbol σ consists of four conditions on a 3×3 matrix (π_i^a) :

$$(1.3) \quad \begin{aligned} 0 &= \pi_3^2 - \pi_2^3, \\ 0 &= \pi_3^1 - \pi_1^3, \\ 0 &= \pi_2^2 - \pi_1^3, \\ 0 &= \pi_2^1 - \pi_1^2 \end{aligned}$$

1(a). Rank-One Ideal. The fundamental theorem of linear algebra states that any homomorphism $\pi \in W \otimes V^*$ has a well-defined rank. Thus, for any tableau $A \subset W \otimes V^*$, we could ask how $\text{rank}(\pi)$ varies across $\pi \in A$. For our purposes, the most interesting¹ case is $\text{rank}(\pi) = 1$.

The space $W \otimes V^*$ admits the *Rank-One Ideal*, \mathcal{R} , which is irreducible and generated by all 2×2 minors $\{0 = \pi_i^a \pi_j^b - \pi_j^a \pi_i^b\}$ in any basis. This is a homogeneous ideal, so we may consider the rank-one cone in vector space or the rank-one variety in projective space. (The vertex of the rank-one cone is the rank-0 matrix.)

For any A , consider the ideal $A^\perp + \mathcal{R}$, which defines $\mathcal{C} \subset A$ as the variety $\mathcal{C} = A \cap \text{Var}(\mathcal{R})$. The variety \mathcal{C} is the set of matrices in A that are also rank-one; it is a linear section of the rank-one cone defined by \mathcal{R} .

In the example (1.2), \mathcal{C} can be parametrized as matrices of the form

$$(1.4) \quad \begin{pmatrix} \kappa^4 & \kappa^3 \tau & \kappa^2 \tau^2 \\ \kappa^3 \tau & \kappa^2 \tau^2 & \kappa \tau^3 \\ \kappa^2 \tau^2 & \kappa \tau^3 & \tau^4 \end{pmatrix} = \begin{pmatrix} \kappa^2 \\ \kappa \tau \\ \tau^2 \end{pmatrix} \otimes \begin{pmatrix} \kappa^2 & \kappa \tau & \tau^2 \end{pmatrix},$$

¹There is a good reason that the rank-1 case is most interesting: the varieties of higher-rank matrices are determined algebraically by the varieties of lower-rank matrices, so the geometry of $\text{rank}(\pi)$ across $\pi \in A$ comes down to the rank-1 case.

which can be interpreted as the rational normal Veronese curve²,

$$(1.5) \quad [\kappa^4 : \kappa^3\tau : \kappa^2\tau^2 : \kappa\tau^3 : \tau^4] \cong \mathbb{P}^1 \subset \mathbb{P}^4 \cong \mathbb{P}A.$$

Moreover, the projection of \mathcal{C} to $\mathbb{P}V^*$ is another rational normal curve,

$$(1.6) \quad [\kappa^2 : \kappa\tau : \tau^2] \cong \mathbb{P}^1 \subset \mathbb{P}^2 \cong \mathbb{P}V^*.$$

This toy example plays a crucial role in applications for hyperbolic and hydrodynamically integrable PDEs [FHK09, Smi09].

1(b). Generic Bases. We would like to find a “good” basis in which to express a tableau A and study its properties.

First, an analogy. When studying a single homomorphism $B : \mathbb{C}^r \rightarrow \mathbb{C}^p$, there are various “good” bases to express B . A basis of \mathbb{C}^r is “generic” for B if the first $\text{rank}(B)$ columns are independent. A basis of \mathbb{C}^p is “generic” for B if the first $\text{rank}(B)$ rows of B are independent in that basis. Among the generic bases, we can construct particularly “good” bases for writing B . When B is written in a “good” basis, we say it is in a “normal form,” and the normal form allows us readily to study properties of B . For example:

- Use Gaussian elimination³ to place B in reduced row-echelon form. Then, the rank, kernel, and image of B are immediately apparent. The fundamental theorems in linear algebra depend on this normal form.
- Apply a polar/unitary decomposition to find the singular-value decomposition of B . Then, the norm of B and its action with respect to the Hermitian inner products of \mathbb{C}^r and \mathbb{C}^p are immediately apparent. Important theorems in metric geometry and multivariate statistics depend on this normal form.
- Solve a sequence of eigenvalue problems in the case $p = r$ to find Jordan normal form. Then, the eigenspace structure of B , and the commutative algebra of matrices to which it belongs are immediately apparent. The theory of Lie groups and Lie algebras depends on this normal form.

Given a tableau $A \subset W \otimes V^*$ with symbol σ , we are curious whether we can construct bases that are “good” simultaneously for all homomorphisms in the tableau. This situation is considerably more complicated than the situation of a single homomorphism, and it turns out that it is most important to focus on the symbol maps, but we arrive at a satisfying answer in Section 7. By the above analogy, it is convenient to have a notion of “generic” bases formulated in terms of independence. Here is the first step:

In any bases of V^* and W , the tableau A is a space of $r \times n$ matrices only s of whose entries are linearly independent. That is, in a given basis, we can consider the entries $\pi \mapsto \pi_i^a$ as elements of A^* , just as we think of the components $v \mapsto v^i$ of vectors in V as being linear functions on $v \in \mathbb{R}^n$, using the dual basis of V^* .

Across all bases of V^* , there is a maximum number of independent entries that can occur in column 1; call that number s_1 . (In a measure-zero set of bases of V^* , the number of actual independent entries in the first column may be less than s_1 .) Once those independent entries are accounted for, there is a maximum number s_2 of new independent

² For more on Veronese curves and the related Segre embeddings, see [Har92, Sha94].

³ Algorithmically, this is accomplished using improved Gram-Schmidt or Householder triangularization. See [TB97].

entries that can occur in the second column. (In a measure-zero set of bases of V^* that achieve s_1 in column 1, the number of actual independent entries in columns 1 and 2 may be less than $s_1 + s_2$.) Once those independent entries are accounted for, there is a maximum number s_3 of new independent entries that can occur in column 3. (In a measure-zero set of bases of V^* that achieve $s_1 + s_2$ in columns 1 and 2, the number of actual independent entries in columns 1, 2, and 3 may be less than $s_1 + s_2 + s_3$.) Continuing in this way, we have s_i as the number of new independent entries in the i th column achieved for almost-all bases of V^* . (In a measure-zero set of bases of V^* that achieve $s_1 + s_2 + \cdots + s_{i-1}$ in columns 1 through $i-1$, the number of actual independent entries in columns 1 through i may be less than $s_1 + \cdots + s_i$.) Eventually, for such a basis, there is a column ℓ where we have reached $s_1 + s_2 + \cdots + s_\ell = s$, so there is some maximum column $\ell \leq n$ such that $s_\ell > 0$, where the last independent entry appears. So,

$$(1.7) \quad \begin{aligned} s &= s_1 + s_2 + \cdots + s_\ell + s_{\ell+1} + \cdots + s_n \\ &= s_1 + s_2 + \cdots + s_\ell + 0 + \cdots + 0. \end{aligned}$$

The index ℓ is called the *character* of A , and the number s_ℓ is called the *Cartan integer* of A . The tuple (s_1, \dots, s_ℓ) gives the *Cartan characters* of A . Note that $s_1 \geq s_2 \geq \cdots \geq s_\ell$, since otherwise the maximality condition would have been violated in an earlier column.

Permanently reserve the index ranges

$$(1.8) \quad \begin{aligned} i, j &\in \{1, \dots, \ell, \ell + 1, \dots, n\}, \\ \lambda, \mu &\in \{1, \dots, \ell\}, \\ \varrho, \varsigma &\in \{\ell + 1, \dots, n\}, \text{ and} \\ a, b &\in \{1, \dots, r\} \end{aligned}$$

A basis⁴ $(u^i) = (u^1, \dots, u^n)$ of V^* is called *generic* if its characters achieve the lexicographical maximum value (s_1, s_2, \dots, s_n) . As seen in the previous paragraph, almost all bases of V^* are generic in this sense. Given a basis (u^i) of V^* , a basis⁵ $(z_a) = (z_1, \dots, z_r)$ is called *generic* if the *first* s_i independent entries in column i are independent.

Choose generic a basis $(u^i) = (u^1, \dots, u^n)$ for V^* , and let $(u_i) = (u_1, \dots, u_n)$ be its dual basis for V . Choose a generic basis $(z_a) = (z_1, \dots, z_r)$ for W , and let $(z^a) = (z^1, \dots, z^n)$ be its dual basis for W^* . An element of the tableau is written as

$$(1.9) \quad \pi = \pi_i^a(z_a \otimes u^i) \in W \otimes V^*,$$

and the upper-left entries π_λ^a for $a \leq s_\lambda$ form a basis of A^* .

Because the bases are generic, the symbol map σ can be written as

$$(1.10) \quad \left\{ 0 = \pi_i^a - B_{i,b}^{a,\lambda} \pi_\lambda^b \right\}_{s_i < a}.$$

It is implicit that $B_{i,b}^{a,\lambda} = 0$ if $a \leq s_i$ or $b \geq s_\lambda$ or $i < \lambda$. That is, entries to the lower-right are written as linear combinations of the entries in the upper-left using the coefficients $B_{i,b}^{a,\lambda}$, as in Figure 1.

⁴This notation indicates an ordered basis of co-vectors, not a vector. Each u^i is an element of V^* .

⁵This notation indicates an ordered basis for vectors, not a co-vector. Each z_a is an element of W .

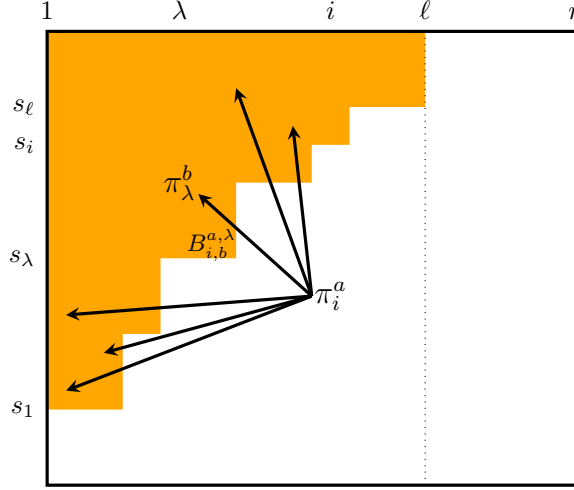


FIGURE 1. A tableau in generic bases. Image from [Smi15].

Consider the example (1.3), which is not written in generic bases. If we exchange columns $2 \leftrightarrow 3$ and rows $1 \leftrightarrow 3$, then it becomes generic with $(s_1, s_2, s_3) = (3, 2, 0)$, seen here:

$$(1.11) \quad \left\{ \begin{pmatrix} \alpha_2 & \alpha_4 & \alpha_3 \\ \alpha_1 & \alpha_3 & \alpha_2 \\ \alpha_0 & \alpha_2 & \alpha_1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \pi_1^1 & \pi_2^1 & \pi_2^2 \\ \pi_1^2 & \pi_2^2 & \pi_1^1 \\ \pi_1^3 & \pi_1^1 & \pi_1^2 \end{pmatrix} \right\}.$$

Equation (1.10) becomes:

$$(1.12) \quad \begin{aligned} 0 &= \pi_2^3 - 1\pi_1^1 - 0\pi_1^2 - 0\pi_1^3 - 0\pi_2^1 - 0\pi_2^2, \\ 0 &= \pi_3^1 - 0\pi_1^1 - 0\pi_1^2 - 0\pi_1^3 - 0\pi_2^1 - 1\pi_2^2, \\ 0 &= \pi_3^2 - 1\pi_1^1 - 0\pi_1^2 - 0\pi_1^3 - 0\pi_2^1 - 0\pi_2^2, \\ 0 &= \pi_3^3 - 0\pi_1^1 - 1\pi_1^2 - 0\pi_1^3 - 0\pi_2^1 - 0\pi_2^2. \end{aligned}$$

One can take the dual perspective, wherein the symbol coefficients $B_{i,b}^{a,\lambda}$ define a map from the upper-left independent entries to the lower-right entries. That is, consider the map

$$(1.13) \quad B \in V^* \otimes V \otimes W \otimes W^* \cong \text{End}(V^*) \otimes \text{End}(W)$$

defined by

$$(1.14) \quad \sum_{a \leq s_i} \delta_i^\lambda \delta_b^a (z_a \otimes z^b) \otimes (u^i \otimes u_\lambda) + \sum_{a > s_i} B_{i,b}^{a,\lambda} (z_a \otimes z^b) \otimes (u^i \otimes u_\lambda).$$

Equation (1.14) is the formal inclusion $A \rightarrow W \otimes V^*$ in the defining exact sequence (1.1).

By fixing $\varphi \in V^*$ and $v \in V$, we obtain an endomorphism $B(\varphi)(v) : W \rightarrow W$ defined by the column relations of (π_i^a) , as in Figure 2. We use the shorthand B_i^λ for $B(u^\lambda)(u_i)$,

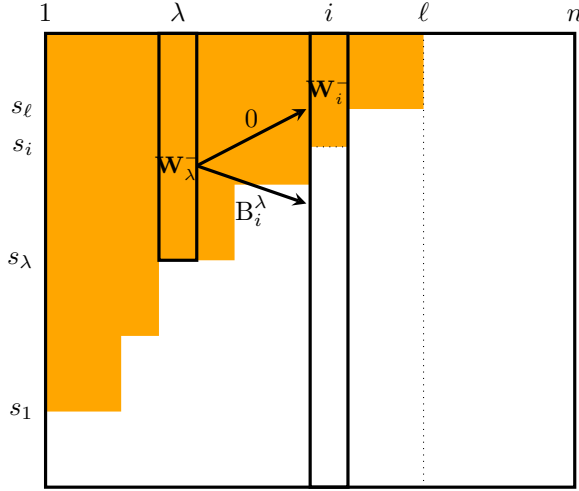


FIGURE 2. The map B_i^λ for a tableau in generic bases. Image from [Smi15].

but note that this is not quite the same as $B_{i,b}^{a,\lambda} z_a \otimes z^b$ because of the identity term in Equation (1.14).

For the example (1.11)–(1.12), the maps $B_i^\lambda : W \rightarrow W$ are:

$$(1.15) \quad \begin{aligned} B_1^1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & B_2^1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & B_3^1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ B_2^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & B_3^2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So, if $\varphi = \varphi_i u^i \in V^*$ and $v = v^j u_j \in V$, the endomorphism $B(\varphi)(v) : W \rightarrow W$ is

$$(1.16) \quad B(\varphi)(v) = \begin{pmatrix} \varphi_1 v^1 + \varphi_2 v^2 & \varphi_2 v^3 & 0 \\ \varphi_1 v^3 & \varphi_1 v^1 + \varphi_2 v^2 & 0 \\ \varphi_1 v^2 & \varphi_1 v^3 & \varphi_1 v^1 \end{pmatrix}.$$

Using our generic basis (u_i) for V and its dual basis (u^i) for V^* , define decompositions $V = U \oplus Y$ and $V^* = Y^\perp \oplus U^\perp$ using our index convention (1.8) as follows:

$$\begin{aligned}
V &= \langle u_1, \dots, u_\ell, u_{\ell+1}, \dots, u_n \rangle = \langle u_i \rangle \\
U &= \langle u_1, \dots, u_\ell \rangle = \langle u_\lambda \rangle \\
Y &= \langle u_{\ell+1}, \dots, u_n \rangle = \langle u_\rho \rangle,
\end{aligned}
\tag{1.17}$$

and

$$\begin{aligned}
V^* &= \langle u^1, \dots, u^\ell, u^{\ell+1}, \dots, u^n \rangle = \langle u^i \rangle \\
U^* &\cong Y^\perp = \langle u^1, \dots, u^\ell \rangle = \langle u^\lambda \rangle \\
Y^* &\cong U^\perp = \langle u^{\ell+1}, \dots, u^n \rangle = \langle u^\varrho \rangle.
\end{aligned}
\tag{1.18}$$

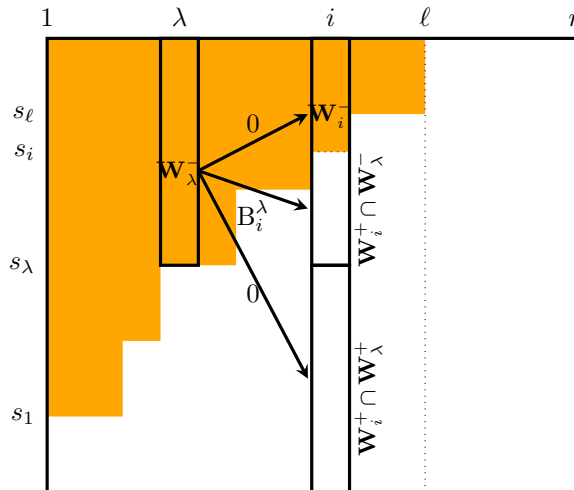


FIGURE 3. The map B_i^λ for an endovolutive tableau.

The isomorphisms $U^* \cong Y^\perp$ and $Y^* \cong U^\perp$ depend on the basis; they are unnatural but sometimes useful.

It is apparent from (1.14) that $B(\varphi) = B(\tilde{\varphi})$ if $\varphi - \tilde{\varphi} \in U^\perp$; that is if $\varphi_\ell = \tilde{\varphi}_\ell$ for all $\ell \geq \ell + 1$, so we usually consider $B(\varphi)$ only for $\varphi \in Y^\perp$.

Thus, in generic bases, we have a collection B_i^λ of endomorphisms of W . For our purposes of constructing a normal form, a “good” basis is one which makes the endomorphisms B_i^λ as structurally similar as possible. Section 1(c) imposes additional conditions on the images of these endomorphisms for this purpose.

1(c). Endovolutive Tableaux. Suppose (u^i) and (z_a) are generic bases for A . For any i , define a decomposition $W = \mathbf{W}_i^- \oplus \mathbf{W}_i^+$ by

$$\begin{aligned}
W &= \langle z_1, \dots, z_{s_i}, z_{s_i+1}, \dots, z_n \rangle = \langle z_a \rangle \\
\mathbf{W}_i^- &= \langle z_1, \dots, z_{s_i} \rangle \\
\mathbf{W}_i^+ &= \langle z_{s_i+1}, \dots, z_n \rangle
\end{aligned}
\tag{1.19}$$

By (1.14), the map $B_i^\lambda : W \rightarrow W$ has support $\mathbf{W}_\lambda^- \subset W$, and its image lies in $\mathbf{W}_i^+ \subset W$.

More generally, for any $\varphi \in V^*$, let $\mathbf{W}^-(\varphi) = \mathbf{W}_{\underline{\lambda}}^-$ and $\mathbf{W}^+(\varphi) = \mathbf{W}_{\underline{\lambda}}^+$, where $\underline{\lambda}$ is the minimum index such that $\varphi_{\lambda} \neq 0$. (For general φ , we have $\dim \mathbf{W}^-(\varphi) = s_1$.)

A tableau A expressed in bases (u^i) and (z_a) is called *endovolutive*⁶ if $B_{i,b}^{a,\lambda} = 0$ for all $a > s_\lambda$. That is, endovolutive means that B_i^λ is an endomorphism of $\mathbf{W}_\lambda^- \subset W$, as in Figure 3.

⁶The term *endovolution* was coined in [Smi15], but the phenomenon was described previously in [BCG⁺90, Chapter IV§5], [Yan87], and it is certainly familiar to anyone who was manipulated tableaux of linear Pfaffian systems.

When considering endovolutive tableaux, it is useful to arrange the symbol endomorphisms as an $\ell \times n$ array of $r \times r$ matrices:

$$(1.20) \quad \begin{bmatrix} I_{s_1} & B_2^1 & B_3^1 & B_4^1 & \cdots & B_\ell^1 & \cdots & B_n^1 \\ 0 & I_{s_2} & B_3^2 & B_4^2 & \cdots & B_\ell^2 & \cdots & B_n^2 \\ 0 & 0 & I_{s_3} & B_4^3 & \cdots & B_\ell^3 & \cdots & B_n^3 \\ 0 & 0 & 0 & I_{s_4} & \cdots & B_\ell^4 & \cdots & B_n^4 \\ & & & & \ddots & & B_i^\lambda & \vdots \\ 0 & 0 & 0 & 0 & 0 & I_{s_\ell} & \cdots & B_n^\ell \end{bmatrix}$$

In (1.20), endovolativity means that each $r \times r$ sub-matrix in row λ is 0 outside the upper-left $s_\lambda \times s_\lambda$ part. This is how our example was expressed in (1.15). Note that (1.15) is endovolutive because $s_2 = 2$ and B_3^2 has entries only in the upper-left 2×2 part.

If a tableau is endovolutive in certain bases for W and V^* , then it is also endovolutive under any upper-triangular change-of-basis for $u^i \mapsto g_j^i u^j$. Under such a basis change, the columns of (π_i^a) are linear combinations of the ones to their right, and the sub-matrices in (1.20) change by the corresponding conjugation. Endovolativity is a property of the flag generated by the basis of V^* .

1(d). Mutual Eigenvectors and Rank. For endovolutive bases, each B_i^λ is an endomorphism of a particular vector space, so it is sensible to consider an eigenvector problem for these maps: For any λ , let

$$(1.21) \quad \mathbf{W}^1(u^\lambda) = \left\{ w \in \mathbf{W}_\lambda^- : B_\mu^\lambda w = \delta_\mu^\lambda w, \forall \mu \leq \ell \right\}.$$

That is, we want to find the vectors that are preserved by $B_\lambda^\lambda = I_{s_\lambda}$ but are annihilated by all B_μ^λ for $\mu \neq \lambda$. More generally, let

$$(1.22) \quad \mathbf{W}^1(\varphi) = \left\{ w \in \mathbf{W}^-(\varphi) : \left(\sum_\lambda \varphi_\lambda B_\mu^\lambda - \varphi_\mu I \right) w = 0, \forall \mu \leq \ell \right\}.$$

Equation (1.22) can be rewritten as a mutual eigenvector problem on the ℓ endomorphisms $B(\varphi)(u_1), \dots, B(\varphi)(u_\ell)$:

$$(1.23) \quad \mathbf{W}^1(\varphi) = \left\{ w \in \mathbf{W}^-(\varphi) : B(\varphi)(u_\mu) w = \varphi_\mu w, \forall \mu \leq \ell \right\}.$$

Alternatively, because $B_\mu^\mu = I_{s_\mu}$, equation (1.22) says that $B(\varphi)(\cdot)w$ is rank-one when restricted to Y^\perp , so we can rewrite it as

$$(1.24) \quad \mathbf{W}^1(\varphi) = \left\{ w \in \mathbf{W}^-(\varphi) : w \otimes \varphi + J_\varphi^a(z_a \otimes u^\varphi) \in A_e, \exists J \in W \otimes U^\perp \right\}.$$

This space is the focus of [Gui68], and it plays an important part in our story. Unlike $\mathbf{W}^-(\varphi)$, its definition does not depend on the basis; its definition depends only on the splitting $V = U \oplus Y$. Its dimension is an important invariant:

LEMMA 1.25. *Suppose that the tableau A admits endovolutive bases. For generic φ , $\dim \mathbf{W}^1(\varphi) = s_\ell$.*

Lemma 1.25 is the result of a quick rank computation using (1.22)–(1.23). See [Smi15].

Our “good” basis and normal form will be built on the requirement that the maps B_i^λ commute on certain combinations of these spaces (and thus the maps share generalized eigenspaces and Jordan-block normal form there). That is, we are aiming for something like the commutative subalgebras seen in [Ger61] and [GS00]. Endovolutivity allows surprisingly direct computation of the desired conditions. For more detail on endovolutivity, see [Smi15] and the references therein. We return to this topic in Section 5, but before that we must introduce the geometry of subspaces.

2. Grassmann and Universal Bundles

The Grassmann variety is the set $\text{Gr}_n(X)$ of n -planes in an $(n+r)$ -dimensional vector space X . It is a smooth projective variety and a smooth manifold of dimension nr . An n -plane $e \in \text{Gr}_n(X)$ is called an *element*.

2(a). Tangent and Arctangent. Depending on one’s favorite notation, there are several ways to see that the tangent space of $\text{Gr}_n(X)$ at e is $(X/e) \otimes e^*$.

First, for any $e \in \text{Gr}_n(X)$, choose a basis⁷ $(u_i) = (u_1, \dots, u_n)$ for e , and choose $(z_a) = (z_1, \dots, z_r)$ so as to complete a basis of the entire vector space X . Any n -plane \tilde{e} near e admits a basis $(\tilde{u}_i) = (\tilde{u}_1, \dots, \tilde{u}_n)$ that we may assume is related by a matrix in reduced column-echelon form:

$$(2.1) \quad (\tilde{u}_1 \quad \dots \quad \tilde{u}_n) = (u_1 \quad \dots \quad u_n \quad z_1 \quad \dots \quad z_r) \begin{pmatrix} 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \\ K_1^1 & \dots & K_n^1 \\ \vdots & \ddots & \vdots \\ K_1^r & \dots & K_n^r \end{pmatrix}$$

More succinctly, using the summation convention:

$$(2.2) \quad \tilde{u}_i = u_i + z_a K_i^a = u_j \delta_i^j + z_a K_i^a.$$

That is, (\tilde{u}_i) and (u_i) are related by an $(n+r) \times n$ matrix of rank n whose image $\langle \tilde{u}_1, \dots, \tilde{u}_n \rangle = \tilde{e}$ is determined uniquely by the $r \times n$ submatrix (K_i^a) . In this sense, $T_e \text{Gr}_n(X)$ is isomorphic to the space of $r \times n$ matrices, which is isomorphic to $(X/e) \otimes e^*$. This is easy and computational, but this isomorphism is not *natural* for an abstract vector space (without metric) because it depends on a choice of splitting $X = e \oplus (X/e)$ by choosing the complementary basis (z_a) .

Alternatively, to see $T_e \text{Gr}_n(X) = (X/e) \otimes e^*$ and avoid splitting, we can use the dual⁸ short-exact sequences

$$(2.3) \quad \begin{aligned} 0 &\rightarrow e \rightarrow X \rightarrow X/e \rightarrow 0, \\ 0 &\rightarrow e^\perp \rightarrow X^* \rightarrow e^* \rightarrow 0. \end{aligned}$$

⁷This notation indicates an ordered basis of vectors, not a co-vector. Each u_i is a vector in X .

⁸Recall that $(X/e)^*$ is canonically isomorphic to e^\perp : If $[v] = \{u + e\} \in X/e$, then $\varphi([v]) = \varphi(v) + 0$ is well-defined for all $\varphi \in e^\perp$.

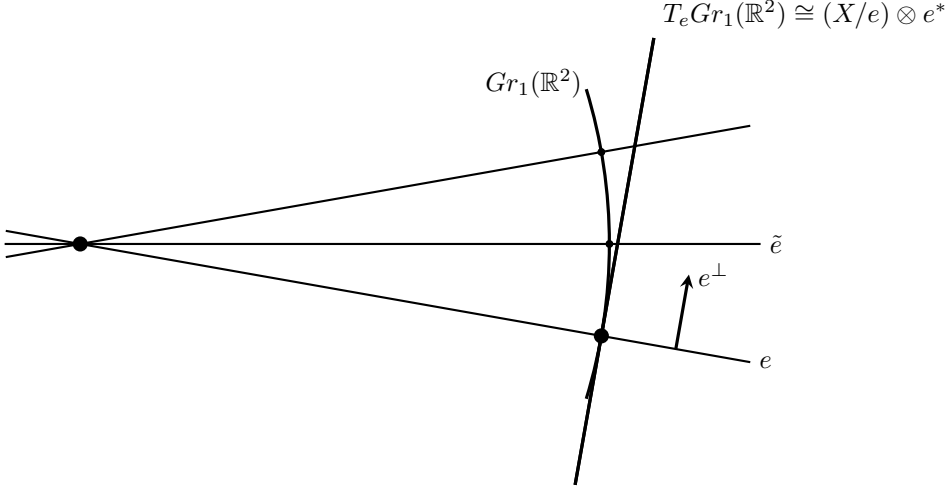


FIGURE 4. From e , identify a nearby line \tilde{e} in \mathbb{R}^2 with a relative angle. The map from $T_e \text{Gr}_1(\mathbb{R}^2) \cong (-\infty, \infty)$ to the neighborhood $(-\pi/2, \pi/2)$ of e in $\text{Gr}_1(\mathbb{R}^2)$ is \arctan_e . Its inverse is \tan_e .

Choose any basis $(\theta^a) = (\theta^1, \dots, \theta^r)$ of the annihilator space $e^\perp = (X/e)^*$, and let $(z_a) = (z_1, \dots, z_r)$ be the corresponding dual basis of (X/e) . Then, we may take the coefficients K_i^a of

$$(2.4) \quad K = z_a \otimes K_i^a(\tilde{e}) = z_a \otimes \theta^a(\tilde{u}_i) \in (X/e) \otimes e^*$$

as nr coordinates on $T_e \text{Gr}_n(X)$; that is, K_i^a gives a basis of $T_e^* \text{Gr}_n(X)$.

More abstractly, an explicit choice of bases (u_i) for e and (θ^a) for e^\perp is unnecessary. Instead, we need only the abstract homomorphism $K \in (X/e) \otimes e^*$, because the space⁹

$$(2.5) \quad \tilde{e} = \langle \tilde{u}_i \rangle = \langle u_i + K(u_i) \rangle = \langle v + K(v) : v \in e \rangle$$

is invariant under $GL(n)$ transformations on (u_i) and (\tilde{u}_i) as well as $GL(r)$ transformations on θ .

As in Figure 4, the derivative map $\text{Gr}_n(X) \rightarrow (X/e) \otimes e^*$ near e is a multidimensional generalization of the tangent function, so the inverse map¹⁰ is written

$$(2.6) \quad \arctan_e : (X/e) \otimes e^* \rightarrow \text{Gr}_n(X).$$

The reader is encouraged to read [MS74, §5] and [KN63] and to search for the terms *Plücker embedding* and *Stiefel manifold* for more detail on this subject.

REMARK 2.7. Notice that any linear subspace of $(X/e) \otimes e^*$ is a tableau in the sense of Section 1. In some sense, it is the *only* example, as arbitrary V and W could be studied

⁹Note that $v + K(v)$ is not well-defined in X for any particular $v \in e$, but the span over all such v is well-defined.

¹⁰The map \arctan_e is analogous to exponential map $\exp_p : T_p M \rightarrow M$ from Riemannian geometry or Lie group representation theory, except that this description of \arctan_e does not make explicit use of a metric or group structure.

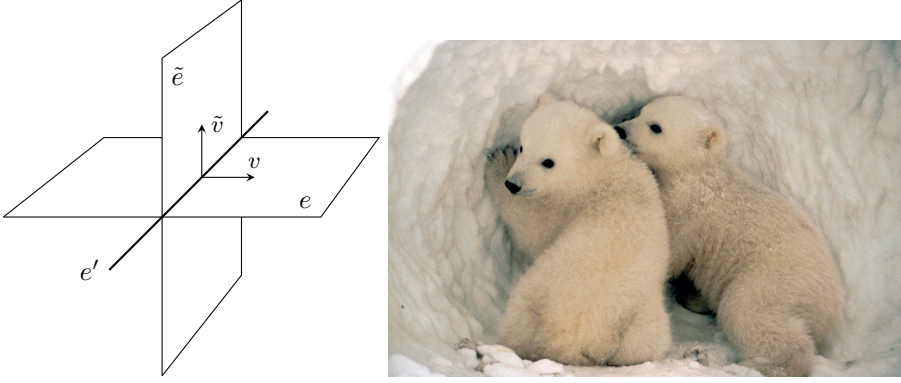


FIGURE 5. Polar pairs.

by setting $X = V \oplus W$ and $e = V + 0$. Moreover, any smooth submanifold $Z \subset \text{Gr}_n(X)$ with tangent space $T_e Z \subset T_e \text{Gr}(X)$ at $e \in Z$ gives $T_e Z$ as a tableau in $(X/e) \otimes e^*$. This observation is the heart of the entire subject of exterior differential systems, and it reappears forcefully in Section 4.

2(b). Polar pairs. Suppose that $e, \tilde{e} \in \text{Gr}_n(X)$, and that they share a hyperplane. That is, suppose $e' = e \cap \tilde{e}$ and $\dim e' = n - 1$. We call the n -planes e and \tilde{e} *polar pairs* because they are both polar extensions¹¹ of e' .

If \tilde{e} is near e in the sense that $\tilde{e} = \arctan_e(K)$ for some K . Let (u_1, \dots, u_{n-1}) be a basis for e' , and extend that basis to a basis (u_1, \dots, u_{n-1}, v) for e and to a basis $(u_1, \dots, u_{n-1}, \tilde{v})$ for \tilde{e} . Writing (2.2) in this case, it is apparent that only the n th column of (K_i^a) is nonzero. That is, the tangent homomorphism $K \in (X/e) \otimes e^*$ is rank-1. (It cannot be rank-0 unless $e = \tilde{e}$.)

Conversely, suppose that $K \in (X/e) \otimes e^*$ is rank-1, and let $\tilde{e} = \arctan_e(K)$. Let $e' = \ker K \subseteq e$, which is a subspace of e of dimension $n-1$. Any line in e' is preserved by the map $e \rightarrow X$ defined by the matrix in (2.2); hence, $e' \subseteq \tilde{e}$.

One can see immediately that this generalizes by replacing 1 with any rank k to obtain a Grassmannian version of the rank-nullity theorem

LEMMA 2.8. *If $e \in \text{Gr}_n(X)$ and $\tilde{e} = \arctan_e(K)$ and $\text{rank}(K) = k$, then*

$$\dim(e \cap \tilde{e}) = n - k.$$

Next, we can generalize Lemma 2.8 by dropping the assumption of nearness, in which case it is possible that K fails to be unique. This non-uniqueness is the same phenomenon as the failure of injectivity at large distances for the exponential map in Riemannian geometry. For any $e \in \text{Gr}_n(X)$, let

$$(2.9) \quad \text{Pol}_k(e) = \{\tilde{e} \in \text{Gr}_n(X) : \dim(\tilde{e} \cap e) = n - k\}.$$

Note that $\text{Pol}_k(e)$ is nonempty if and only if $k \leq r$, because $n + k = \dim(e + \tilde{e}) \leq n + r$.

¹¹This is a classical terminology that reappears in Section 6(a).

LEMMA 2.10. *Fix $e \in \text{Gr}_n(X)$ and $\tilde{e} \in \text{Pol}_k(e)$. The space $(\tilde{e} \cap e)^\perp / e^\perp \subset X^* / e^\perp = e^*$ has dimension k . The space $\tilde{e}/e \subset X/e$ also has dimension k . This yields the incidence correspondence in Figure 6.*

Again the proof is simply a repeated use of the rank-nullity theorem for the short-exact sequences (2.3).

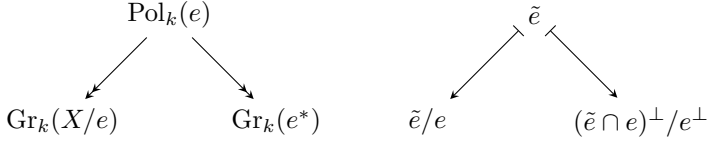


FIGURE 6. The incidence correspondence of polar pairs e and \tilde{e} .

Now, reconsider the case $k = 1$. Then each $\tilde{e} \in \text{Pol}_1(e)$ yields a hyperplane $e' = \tilde{e} \cap e$. The right image $(e')^\perp / e^\perp$ in Figure 6 is some line $[\xi] \in \mathbb{P}e^*$. The left image \tilde{e}/e is some line $[w] \in \mathbb{P}(X/e)$. So, each $\tilde{e} \in \text{Pol}_1(e)$ yields a rank-one projective homomorphism $[w] \otimes [\xi] \in \mathbb{P}((X/e) \otimes e^*)$. Any element of $\mathbb{P}((X/e) \otimes e^*)$ could be obtained this way by appropriate choice of \tilde{e} .

We can write $[w] \otimes [\xi]$ like this: Let $(\omega^1, \dots, \omega^n, \theta^1, \dots, \theta^r)$ be a basis for X^* such that $e = \ker\{\theta^1, \dots, \theta^r\}$ and $e' = \ker\{\theta^1, \dots, \theta^r, \xi\}$ for some $\xi = \xi_i \omega^i$. Then, $\tilde{e} = \ker\{\tilde{\theta}^1, \dots, \tilde{\theta}^r\}$ for some $\tilde{\theta}^a = J_b^a \theta^b + K_i^a \omega^i$. Because $e' \subset \tilde{e}$, it must be that

$$\begin{aligned}
 \tilde{\theta}^a &\equiv 0 \pmod{\{\theta^c, \xi\}}, \text{ so} \\
 J_b^a \theta^b + K_i^a \omega^i &\equiv 0 \pmod{\{\theta^c, \xi\}}, \text{ so} \\
 K_i^a \omega^i &\equiv 0 \pmod{\{\theta^c, \xi\}}, \text{ so} \\
 K_i^a \omega^i &\equiv 0 \pmod{\{\xi\}}.
 \end{aligned}
 \tag{2.11}$$

Hence, each $K_i^a \omega^i$ is a multiple of ξ ; call it $w^a \xi$. (Note that $w^a = 0$ for all a if and only if $\tilde{e} = e$, which contradicts our assumption $\dim e' = n - 1$.) We can use this fact to build a rank-one homomorphism: Let (z_a) be the basis of X/e dual to (θ^a) . Let (ω^i) also denote the basis of $e^* = X^* / e^\perp$ induced by $\omega^i \in X^*$, so that $\xi \in e^*$ also denotes the image of $\xi \in X^*$. Let $w = w^a z_a$. Then the induced homomorphism

$$K = z_a \otimes K_i^a \omega^i = z_a \otimes w^a \xi = w \otimes \xi \in (X/e) \otimes e^* \tag{2.12}$$

has rank 1. Each of w and ξ is defined up to scale, so K is well-defined only up to scale, $[K] = \mathbb{P}((X/e) \otimes e^*)$.

It may be that \tilde{e} lies outside the image where the map \arctan_e is injective. How then do we interpret K ? For some vectors v and w , we may write $e = e' + \langle v \rangle$ and $\tilde{e} = e' + \langle w \rangle$ and define a curve

$$e_\tau = e' + \langle (1 - \tau)v + \tau w \rangle \tag{2.13}$$

in $\text{Gr}_n(X)$ with $e_0 = e$ and $e_1 = \tilde{e}$. Note that $e_\tau \in \text{Pol}_1(e)$ for all $0 < \tau \leq 1$, and by Lemma 2.10 we know e_τ / e is the line $[\tau w]$, and $(e_\tau \cap e)^\perp / e^\perp$ is a line $[\xi]$ that is constant versus τ . Hence, all such e_τ have the same representative rank-1 homomorphism,

$[w] \otimes [\xi] = [K]$. Even if $\tilde{e} = e_1$ lies outside the image of \arctan_e , all e_τ lie inside the image of \arctan_e for an open ray of sufficiently small τ . Since K is defined only to scale, the projective homomorphism $w \otimes \xi$ is shared by all those e_τ . So, the image $\arctan_e(w \otimes \xi)$ contains an open set of $\{e_\tau\}$ where $e_\tau \cap e = e'$.

In Figure 5, e_τ is the family obtained by rotating from e toward \tilde{e} about the axis e' through an angle $\arctan(\frac{\tau}{1-\tau})$. In other words, we have

$$(2.14) \quad e = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \text{ and } \tilde{e} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$

so we can write for all $0 \leq \tau < 1$,

$$(2.15) \quad e_\tau = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1-\tau \\ \tau \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\tau}{1-\tau} \end{pmatrix} \right\rangle.$$

Overall, we have learned that any $k = 1$ polar pair of e is represented by a line of rank-one matrices in $T_e \text{Gr}_n(X)$, like this:

LEMMA 2.16. *For any line $w \otimes \xi \in \mathbb{P}((X/e) \otimes e^*)$, there exists a ray of $K \in T_e \text{Gr}_n(X)$ representing $w \otimes \xi$ such that each $\tilde{e} = \arctan_e(K)$ lies in $\text{Pol}_1(e)$. That is, $w \otimes \xi$ is represented by $\text{Pol}_1(e)$ in any open set of $\text{Gr}_n(X)$ containing e . This also holds for any linear subspace of $T_e \text{Gr}_n(X)$ and corresponding submanifold of $\text{Gr}_n(X)$.*

This is sufficient for our purposes, but those seeking a more detailed understanding of polar pairs are encouraged to investigate *Schubert varieties*—for example in [Rob12]—and the other outgrowths of Hilbert’s 15th problem.

2(c). The Tautological Bundle. Soon, we will consider algebraic equations defined on e^* . To facilitate this, for any $e \in \text{Gr}_n(X)$, we consider the complexified projective space $\mathbb{X} = \mathbb{P}X \otimes \mathbb{C}$ and its subspace $\mathbb{P}e \otimes \mathbb{C}$. For standard complex projective space, we write \mathbb{P}^d for $\mathbb{C}\mathbb{P}^d = \mathbb{P}(\mathbb{C}^{d+1})$. That is, $\mathbb{X} \cong \mathbb{P}^{n+r-1}$, and $\mathbb{P}e \otimes \mathbb{C} \cong \mathbb{P}^{n-1}$.

If we consider all such spaces across all e simultaneously, we obtain the *tautological bundle*¹² γ over $\text{Gr}_n(X)$ with fiber

$$(2.17) \quad \gamma_e = \mathbb{P}e \otimes \mathbb{C}, \quad \forall e \in \text{Gr}_n(X),$$

and its dual bundle γ^* over $\text{Gr}_n(X)$ with fiber

$$(2.18) \quad \gamma_e^* = \mathbb{P}e^* \otimes \mathbb{C}, \quad \forall e \in \text{Gr}_n(X),$$

and its annihilator bundle γ^\perp over $\text{Gr}_n(X)$ with fiber

$$(2.19) \quad \gamma_e^\perp = \mathbb{P}e^\perp \otimes \mathbb{C}, \quad \forall e \in \text{Gr}_n(X),$$

and its cokernel bundle \mathbb{X}/γ over $\text{Gr}_n(X)$ with fiber

$$(2.20) \quad (\mathbb{X}/\gamma)_e = \mathbb{P}(X/e) \otimes \mathbb{C}, \quad \forall e \in \text{Gr}_n(X).$$

See Figure 7.

¹² These are also called universal bundles or canonical bundles. They are analogous to the sheaves $\mathcal{O}(-1)$ and $\mathcal{O}(1)$, respectively, for varieties in projective space.

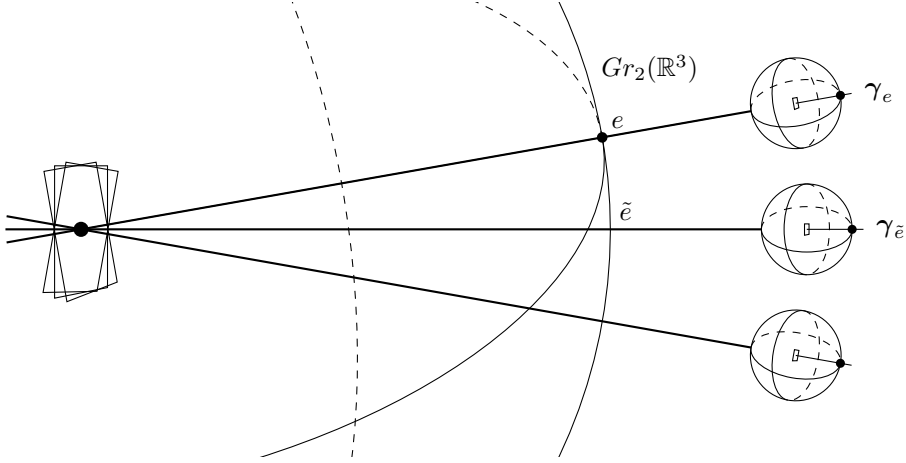


FIGURE 7. A cartoon of the tautological bundle, γ . Here e is a real 2-plane in \mathbb{R}^3 , which can be represented by a line because $Gr_2(\mathbb{R}^3) \cong \mathbb{P}(\mathbb{R}^{3*})$. Each $\gamma_e \cong \mathbb{P}(\mathbb{R}^2) \otimes \mathbb{C} = \mathbb{P}^1$ is a Riemann sphere. Thus, γ is depicted as a bundle of 2-spheres over a hemisphere.

There is a dual pair of short exact sequences of projective bundles, analogous to (2.3).

$$(2.21) \quad \begin{aligned} 0 \rightarrow \gamma_e \rightarrow \mathbb{X} \rightarrow (\mathbb{X}/\gamma)_e \rightarrow 0, \\ 0 \rightarrow \gamma_e^\perp \rightarrow \mathbb{X}^* \rightarrow \gamma_e^* \rightarrow 0. \end{aligned}$$

Hence, the projective tangent bundle $\mathbb{P}T \operatorname{Gr}(X) \otimes \mathbb{C}$ is isomorphic (naturally) to $(\mathbb{X}/\gamma) \otimes \gamma^*$. If we choose a section of these sequences, then we can use the dual bases to establish an (unnatural) decomposition $\mathbb{P}X \otimes \mathbb{C} \cong \gamma_e \oplus (\mathbb{X}/\gamma)_e$ for any e .

One can also consider the frame¹³ bundle \mathcal{F}_γ over $\operatorname{Gr}_n(X)$ associated to γ , whose fiber is all linear isomorphisms

$$(2.22) \quad \mathcal{F}_{\gamma,e} = \{(u^i) : \gamma_e \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } \gamma_e^*\} \cong PGL(n),$$

and the coframe bundle \mathcal{F}_{γ^*} over $\operatorname{Gr}_n(X)$ associated to γ^* , whose fiber is all linear isomorphisms

$$(2.23) \quad \mathcal{F}_{\gamma^*,e} = \{(u_i) : \gamma_e^* \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } \gamma_e\} \cong PGL(n).$$

To write homogeneous complex-algebraic ideals on γ_e^* that vary across $e \in \operatorname{Gr}_n(X)$, the appropriate ring is therefore

$$(2.24) \quad S = C^\infty(\operatorname{Gr}_n(X))[u_1, \dots, u_n], \text{ for } (u_i) \text{ a section of } \mathcal{F}_{\gamma^*}.$$

¹³Some authors might flip the names of the frame and coframe bundles. I tend to choose this notation because the frame bundle is covariant with diffeomorphisms on the base space, and only contravariant objects get a “co-” prefix. The jargon for duality is always frustrating.

Part II

PDEs on Manifolds

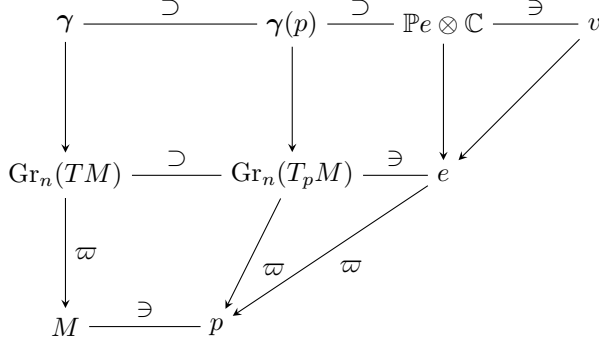


FIGURE 8. Tautological bundles over Grassmann bundles over manifolds. Vertical arrows are bundle projections.

In this part, we build bundles whose fibers are the structures seen in Part I. This produces a satisfying language for describing a system of PDEs on a manifold in Section 4.

3. Bundles upon Bundles

If M is a smooth manifold of dimension $m = n + r$, then we can form the smooth bundle $\text{Gr}_n(TM)$ with fiber $\text{Gr}_n(T_p M)$. Let $\varpi : \text{Gr}_n(TM) \rightarrow M$ denote the bundle projection.

Because (2.3) holds across the bundle for $X = T_p M$, any local section of $\text{Gr}_n(TM)$ can be described by choosing its annihilator section of $\text{Gr}_r(T^*M)$, and vice-versa.

For every $p \in M$, its Grassmann fiber $\text{Gr}_n(T_p M)$ has a tautological bundle $\gamma(p)$ with fiber $\gamma_e(p) = \mathbb{P}e \otimes \mathbb{C}$. The total space $\text{Gr}_n(TM)$ is a manifold in its own right. Hence, we may consider γ as a bundle over the manifold $\text{Gr}_n(TM)$, which is itself a bundle over M . In other words, we can reinterpret Section 2(c), replacing the vector space X with the projective bundle \mathbb{X} over $\text{Gr}_n(TM)$ which has fiber $\mathbb{X}_e = \mathbb{P}T_p M \otimes \mathbb{C}$ at e with $\varpi e = p$. A complete description of some $v \in \gamma$ would be (p, e, v) where $v \in \mathbb{P}e \otimes \mathbb{C}$, and $e \in \text{Gr}_n(T_p M)$, and $p \in M$. See Figure 8. A complete description of some $\varphi \in \gamma^*$ would be (p, e, φ) where $\varphi \in \mathbb{P}e^* \otimes \mathbb{C}$, and $e \in \text{Gr}_n(T_p M)$, and $p \in M$. See Figure 8. The same bundle-wise constructions hold for γ^\perp , (\mathbb{X}/γ) , \mathcal{F}_γ , and \mathcal{F}_{γ^*} from Section 2(c). Extending (2.24) to write homogeneous complex-algebraic ideals on γ_e^* that vary across $e \in \text{Gr}_n(TM)$, the appropriate ring is therefore

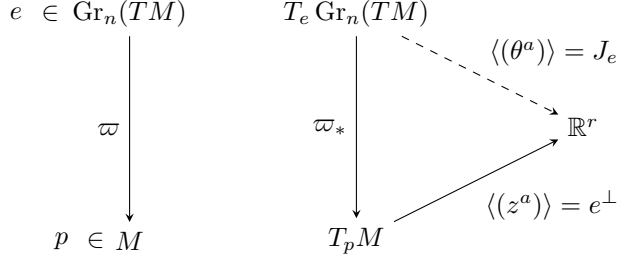
$$(3.1) \quad S = C^\infty(\text{Gr}_n(TM))[u_1, \dots, u_n] \otimes \mathbb{C}, \text{ for } (u_i) \text{ a section of } \mathcal{F}_{\gamma^*}.$$

3(a). The Contact Ideal. For any $e \in \text{Gr}_n(TM)$, consider its annihilator subspace $e^\perp \subset T_p^* M$. There is a corresponding subspace $J_e \subset T_e^* \text{Gr}_n(TM)$, defined as

$$(3.2) \quad J_e = \langle \zeta \circ \varpi_* \text{ such that } \zeta \in e^\perp \rangle = e^\perp \circ \varpi_*.$$

as in Figure 9. If (z^a) is a basis of e^\perp , then we can define a basis of J_e by $\theta^a = z^a \circ \varpi_*$.

The differential ideal $\mathcal{J} \subset \Omega^\bullet(\text{Gr}_n(TM))$ generated by $\langle \theta^a, d\theta^a \rangle$ from J is called the *contact ideal*. Note that, for any (local) section $\epsilon : M \rightarrow \text{Gr}_n(TM)$, the contact ideal

FIGURE 9. Contact forms on the Grassmann bundle of M .

satisfies the universal reproducing property

$$(3.3) \quad \epsilon^*(J) = \epsilon^*(\epsilon^\perp \circ \varpi_*) = \epsilon^\perp \circ \varpi_* \circ \epsilon_* = \epsilon^\perp.$$

Because this property is universal, the subbundle J is a submodule defined globally across $\text{Gr}_n(TM)$ even if topology forces any particular section ϵ to be defined locally.

If one were to choose local coordinates (x^i, y^a) for M and local fiber coordinates (P_i^a) for $\text{Gr}_n(TM)$ near a particular n -plane $e = \ker\{dy^a\}$, then \mathcal{J} is the differential ideal typically written as

$$(3.4) \quad \begin{cases} 0 = \theta^a = dy^a - P_i^a dx^i, \\ 0 = d\theta^a = dP_i^a \wedge dx^i, \end{cases}$$

where the functions P_i^a depend on \tilde{e} in an open neighborhood of e in $\text{Gr}_n(TM)$.

After reading Section 3(b), compare this coordinate description to your favorite definition of jet space, $\mathbb{J}^1(\mathbb{R}^n, \mathbb{R}^r)$. Also, compare the local fiber coordinates P_i^a to the tangent coordinates K_i^a from Section 2(a); when restricting to the fiber over a single basepoint $p \in M$, they are essentially identical. For some highly amusing applications of the contact system, see [Gro86].

3(b). Immersions and Frame Bundles. Fix an immersion $\iota : N \rightarrow M$ with $\dim N = n$. For any $x \in N$ with $\iota(x) = p$, the push-forward derivative has image $\iota_*(T_x N)$, which is an n -dimensional subspace of $T_p M$; hence, $\iota_*(T_x N) \in \text{Gr}_n(TM)$. Define the map $\iota^{(1)} : N \rightarrow \text{Gr}_n(TM)$ by

$$(3.5) \quad \iota^{(1)}(x) = \iota_*(T_x N) = e \in \text{Gr}_n(TM),$$

and note that $\iota = \varpi \circ \iota^{(1)}$, so $\iota_* = \varpi_* \circ \iota_*^{(1)}$.

It is obvious from the definition that $\iota^{(1)}$ is also an immersion. Therefore, we can use it to pull-back the tautological bundle γ^* as defined in Sections 2(c) and 3. Let $\gamma_N^* = \iota^{(1)*}\gamma^*$, which has fiber

$$(3.6) \quad \gamma_{N,x}^* = \gamma_e^*(p) = \mathbb{P}e^* \otimes \mathbb{C} = \mathbb{P}\iota_*(T_x N) \otimes \mathbb{C};$$

that is, γ_N^* is identified with $\mathbb{P}T^*N \otimes \mathbb{C}$ via ι_* . See Figure 10.

The immersion $\iota^{(1)}$ is called the *prolongation* of the immersion ι .

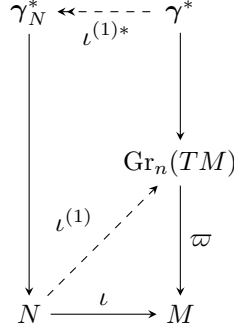


FIGURE 10. The dual tautological bundle γ^* pulls back to form a bundle γ_N^* over an immersed submanifold N .

Now, consider the contact forms $(\theta^a) = (z^a \circ \varpi_*)$ forms from Section 3(a). For all $x \in N$ and all $v \in T_x N$, we have

$$(3.7) \quad \iota^{(1)*}(\theta^a)(v) = \theta^a(\iota_*^{(1)}(v)) = z^a \circ \varpi_* \circ \iota_*^{(1)}(v) = z^a(\iota_*(v)) = 0,$$

which ultimately gives the following lemma:

LEMMA 3.8. *If $\iota : N \rightarrow M$ is an immersion for $\dim N = n$, then $\iota^{(1)*}(\mathcal{J}) = 0$. Conversely, if $\iota' : N \rightarrow \text{Gr}_n(TM)$ is an immersion for $\dim N = n$ satisfying $\iota'^*(\mathcal{J}) = 0$ and such that the image $\iota'_*(T_x N)$ is transverse to the fiber $\ker \varpi_*$ for all $x \in N$, then there is some immersion $\iota : N \rightarrow M$ such that $\iota^{(1)} = \iota'$.*

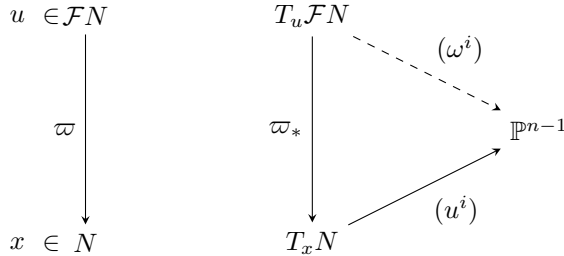


FIGURE 11. Tautological Form of the frame bundle of a manifold N .

Moreover, recall that any manifold N of dimension n admits a projective frame bundle $\varpi : \mathcal{F}N \rightarrow N$ with fiber

$$(3.9) \quad \mathcal{F}_x N = \{(u^i) : T_x N \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } T_x^* N\} \cong PGL(n),$$

The total space $\mathcal{F}N$ admits a tautological¹⁴ 1-form $\omega : \mathcal{F}N \rightarrow \mathbb{P}^{n-1}$ defined by $\omega_u^i = u^i \circ \varpi_*$ as in Figure 11. It is characterized by its universal reproducing property: for any

¹⁴In various references, this 1-form is called the canonical 1-form, the Hilbert 1-form, and the soldering 1-form.

(local) section $\eta : N \rightarrow \mathcal{F}N$:

$$(3.10) \quad \eta^*(\omega^i) = \eta^*(\eta^i \circ \varpi_*) = \eta^i \circ \varpi_* \circ \eta_* = \eta^i,$$

or, more succinctly, $\eta^*(\omega) = \eta$.

Because this property is universal, the 1-form ω is defined globally across $\mathcal{F}N$ even if topology forces any particular 1-form η to be defined locally.

For any local diffeomorphism $f : N \rightarrow \tilde{N}$, there is an induced (covariant) map on the frame bundles $f^\dagger : \mathcal{F}N \rightarrow \mathcal{F}\tilde{N}$ by $f^\dagger : (u^i) \mapsto (u^i) \circ (f_*)^{-1}$. Using the universal property, it is easy to prove this lemma, which shows that diffeomorphisms are characterized by their preservation of the tautological form on the frame bundle:

LEMMA 3.11. *If $f : N \rightarrow \tilde{N}$ is a diffeomorphism, then $(f^\dagger)^*(\tilde{\omega}) = \omega$. Conversely, if $F : \mathcal{F}N \rightarrow \mathcal{F}\tilde{N}$ is $PGL(n)$ -equivariant diffeomorphism such that $F^*(\tilde{\omega}) = \omega$, then there exists a unique diffeomorphism $f : N \rightarrow \tilde{N}$ such that $f^\dagger = F$.*

Combining the universal properties of the \mathcal{J} and ω , we obtain the following theorem telling us what information we can transfer from $\text{Gr}_n(TM)$ to an immersed submanifold:

THEOREM 3.12. *If $\iota : N \rightarrow M$ is a smooth immersion, then*

- $\iota^{(1)*}(\mathcal{J}) = 0$, and
- $\mathcal{F}N = \iota^{(1)*}(\mathcal{F}_\gamma)$.

Conversely, if $\iota' : N \rightarrow \text{Gr}_n(TM)$ is a smooth immersion such that

- $\iota'^*(\mathcal{J}) = 0$, and
- $\mathcal{F}N = \iota'^*(\mathcal{F}_\gamma)$,

then there exists a smooth immersion $\iota : N \rightarrow M$ such that $\iota^{(1)} = \iota'$.

That is, an immersed submanifold satisfies the contact ideal, which is generated differentially by some annihilator 1-forms (θ^a) spanning γ^\perp , and its frame bundle is equipped with tautological 1-forms (ω^i) spanning γ^* .

REMARK 3.13. Note the similarity between the universal property of the contact ideal on the Grassman bundle and the universal property of the tautological 1-form on the frame bundle. Exploitation of this interaction as in Theorem 3.12 has a long and interesting history.

For example, consider the study of a Lie pseudogroup acting on a manifold M . One option is to differentiate the coordinates of M repeatedly using the contact ideal until differential syzygies of the Lie pseudogroup action can be found in prolonged local coordinates, which are then converted to a coordinate-free description using the pseudogroup action. The other option is to work on the frame bundle of M immediately, where any expression on the tautological 1-form automatically invariant, then prolong as necessary to reveal the syzygies. The latter is used often when the Lie pseudogroup arises as equivalence of intrinsic G -structures, and the former is used often when the Lie pseudogroup arises from an extrinsic action on some ambient coordinates. For more on these fascinating and interconnected ideas, I encourage you to read [Cle17], [Olv95], [Val13], and [Gar89]—and the collected works of E. Cartan.

4. Exterior Differential Systems

Let M be a smooth manifold of finite dimension m . An *exterior differential system* [EDS] on M consists of an ideal \mathcal{I} in the total exterior algebra $\Omega^\bullet(M)$ that is differentially closed and finitely generated. *Differentially closed* means that $d\mathcal{I} \subset \mathcal{I}$. *Finitely generated* means that in each degree p , the p -forms in the ideal, $\mathcal{I}_p = \mathcal{I} \cap \Omega^p(M)$, form a finitely generated $C^\infty(M)$ -module. We assume that $\mathcal{I}_0 = 0$; otherwise, one would restrict to a subvariety of M defined by those functions. Optionally, we sometimes specify an independence condition as an n -form $\omega \in \Omega^n(M)$ that is not allowed to vanish on solutions.

REMARK 4.1. Exterior differential systems are defined this way because the term “PDE” or “system of PDEs” is difficult to pin down with geometric precision. Colloquially, “system of PDEs” usually means a finite set of (hopefully, smooth) equations on some local jet space. In Section 2, we explored the geometry of the bundle $\text{Gr}_n(TM)$; recall that the contact system \mathcal{J} on $\text{Gr}_n(TM)$ provides a coordinate-invariant notion of jet space. So, a system of PDEs can be thought of as a collection of equations on jet $\text{Gr}_n(TM)$. Hopefully, those equations are smooth and respect the bundle structure coming from the contact system (otherwise, derivatives misbehave). By virtue of the Plücker embedding $\text{Gr}_n(TM) \rightarrow \mathbb{P} \wedge^n(TM)$, an EDS provides precisely the structure to write an ideal whose variety is a subvariety (in the bundle sense) of $\text{Gr}_n(TM)$. By taking smooth subvarieties, we can apply Remark 2.7 and apply our knowledge of tableaux from Part I to study EDS. Even by this definition, an EDS could be rather wild; however, in many practical applications, it happens that \mathcal{I} is generated by a finite collection of smooth differential forms of homogeneous degree, so one obtains a smooth algebraic variety in local fiber coordinates of $\text{Gr}_n(TM)$.

4(a). Differential Ideals and Integral Elements. To be precise, an *integral element* of \mathcal{I} at $p \in M$ is a linear subspace $e \subset T_p M$ such that $\varphi|_e = 0$ for all $\varphi \in \mathcal{I}_n$. That is, the n -forms in \mathcal{I} provide a collection of functions that cut out a variety, $\text{Var}_n(\mathcal{I}) \subset \text{Gr}_n(TM)$. These functions vary smoothly in M and are homogeneous in the fiber variables.

There is a maximal dimension n for which $\text{Var}_n(\mathcal{I})$ is locally non-empty, which is the case of interest. If an independence condition ω is specified, we also require $\omega|_e \neq 0$, which forces $\text{Var}_n(\mathcal{I})$ to lie in the open subset of $\text{Gr}_n(TM)$ for which that condition holds. (For example, in the case of the contact system, the condition $\omega = dx^1 \wedge \cdots \wedge dx^n \neq 0$ holds in the same neighborhood where (3.4) makes sense.)

Because \mathcal{I}_n is finitely generated by smooth functions, Sard’s theorem guarantees an open, dense subset $\text{Var}_n^o(\mathcal{I}) \subset \text{Var}_n(\mathcal{I})$ defined as the smooth subbundle of $\text{Gr}_n(TM)$ that is cut out smoothly by smooth functions. These are the *Kähler-ordinary* integral elements. A single connected component of $\text{Var}_n^o(\mathcal{I})$ is called $M^{(1)}$, and we allow ourselves to redefine M so that $\varpi : M^{(1)} \rightarrow M$ is a smooth bundle.

Let s denote the dimension of each fiber of the projection $M^{(1)} \rightarrow M$, so $t = nr - s$ is the corresponding codimension of $T_e M_p^{(1)}$ in $T_e \text{Gr}_n(T_p M)$. That is, the projective bundle $A = \ker \varpi_* = TM^{(1)} \subset T \text{Gr}_n(TM)$ is a tableau in the sense of Remark 2.7, as each fiber

$A_e = T_e M_p^{(1)}$ is a linear subspace of $T_e \text{Gr}_n(T_p M)$. Because $M^{(1)}$ is a smooth manifold, we have:

LEMMA 4.2. $K \in A_e$ implies $\arctan_e(K) \in M^{(1)}$ near e .

That is, we have a well-defined vector bundle $A = \ker \varpi_* \subset TM^{(1)}$ over $M^{(1)}$. So that we may apply the results of Section 1, we also restrict ourselves to an open subset of $M^{(1)}$ where the Cartan characters of A_e are constant versus $e \in M^{(1)}$. This is the assumption that $M^{(1)}$ consists of *Kähler-regular* integral elements.

Moreover, to generalize the notation and results of Part I to $M^{(1)}$, define the restricted tautological bundles

$$(4.3) \quad \begin{aligned} V &= \gamma|_{M^{(1)}} = \{\mathbb{P}e \otimes \mathbb{C}\}, \\ V^* &= \gamma^*|_{M^{(1)}} = \{\mathbb{P}e^* \otimes \mathbb{C}\}, \\ W &= (\mathbb{X}/\gamma)|_{M^{(1)}} = \{\mathbb{P}(T_p M/e) \otimes \mathbb{C}\}, \\ V^\perp &= \gamma^\perp|_{M^{(1)}} = \{\mathbb{P}e^\perp \otimes \mathbb{C}\} \end{aligned}$$

Sometimes, it is convenient to think of A as being a complex projective bundle, in which case we consider it to be a subbundle of $W \otimes V^*$.

An *integral manifold* of \mathcal{I} is an immersion $\iota : N \rightarrow M$ such that $\iota^*(\varphi) = 0$ for all $\varphi \in \mathcal{I}$. (If an independence condition ω is specified, we require that $\iota^*(\omega) \neq 0$, too.) When we are considering a particular $M^{(1)} \subset \text{Var}_n(\mathcal{I})$ as above, we say N is an *ordinary* integral manifold provided that $\iota_*(TM) \subset M^{(1)}$. All of the observations from Section 3(b) apply, but $\iota^{(1)}(N)$ lies in the submanifold $M^{(1)}$, and $\iota_*^{(1)}(TN)$ lies in the subbundle A . The overall goal is to construct all ordinary integral manifolds of (M, \mathcal{I}) through the careful study of the geometry of $M^{(1)}$.

4(b). Prolongation and Spencer Cohomology. Suppose that $\iota : N \rightarrow M$ is an ordinary integral manifold of \mathcal{I} . By Theorem 3.12, the 1-forms θ^a spanning J_e must vanish for each $e \in \iota^{(1)}(N)$. The tautological form (ω^i) on \mathcal{F}_τ pulls back to a nondegenerate frame (η^i) on N , since $\iota^{(1)}$ is an immersion.

Therefore, if $\iota^{(1)} : N \rightarrow M^{(1)}$ actually exists, we have

$$(4.4) \quad \begin{aligned} \iota^{(1)*}(\theta^a) &= 0, \\ \iota^{(1)*}(d\theta^a) &= 0, \\ \iota^{(1)*}(\omega^1 \wedge \dots \wedge \omega^n) &= \eta^1 \wedge \dots \wedge \eta^n \neq 0. \end{aligned}$$

However, working on the frame bundle of $M^{(1)}$, these forms satisfy a more general equation, called *Cartan's structure equation*:

$$(4.5) \quad d\theta^a \equiv \pi_i^a \wedge \omega^i + \frac{1}{2} T_{i,j}^a \omega^i \wedge \omega^j, \quad \text{mod } \{\theta^b\}.$$

Using the dual coframe $z_a \leftrightarrow \theta^a$ for $W \leftrightarrow V^\perp$, we can see that $\pi = \pi_i^a(z_a \otimes \omega^i)$ lies in A . In particular, it must be that $\iota^{(1)*}(\pi_i^a) = P_{i,j}^a \eta^j$ for some function $P_{i,j}^a$ that must satisfy $P_{i,j}^a \eta^i \wedge \eta^j = 0$, so $P_{i,j}^a = P_{j,i}^a$. That is, the homomorphism $P \in A \otimes V^*$ lies in the fiber over e of the subbundle

$$(4.6) \quad A \otimes V^* \subset (W \otimes V^*) \otimes V^* = W \otimes (V^* \otimes V^*).$$

Moreover, the existence of an immersion $\iota^{(1)} : N \rightarrow M^{(1)}$ requires that the *torsion* term $w_a T_{i,j}^a \omega^i \wedge \omega^j$ is zero in (4.5); that is, it must be possible to rewrite $\pi_i^a = \pi_i^a + Q_{i,j}^a \omega^j$ for $Q \in A \otimes V^*$ such that any $T_{i,j}^a$ term is absorbed. Note that this absorption of torsion is an algebraic property of the tableau A .

In summary,

LEMMA 4.7. *Let $\delta : A \otimes V^* \rightarrow W \otimes \wedge^2 V^*$ denote the composition of skewing $\otimes^2 V^* \rightarrow \wedge^2 V^*$ and inclusion $A \rightarrow W \otimes V^*$, and write $A^{(1)} = \ker \delta$ and $H^2(A) = \operatorname{coker} \delta$:*

$$(4.8) \quad 0 \rightarrow A^{(1)} \rightarrow A \otimes V^* \xrightarrow{\delta} W \otimes \wedge^2 V^* \rightarrow H^2(A) \rightarrow 0.$$

For any ordinary integral manifold N , the homomorphism P of (4.6) lies in $A^{(1)}$, and the pullback of torsion T is zero in $H^2(A)$.

Writing δ in a chosen coframe, it is easy to check that

$$(4.9) \quad \dim A^{(1)} \leq s_1 + 2s_2 + \cdots + ns_n.$$

The case of equality is considered in Section 5.

The exterior differential system $\mathcal{I}^{(1)}$ on $M^{(1)}$ generated as

$$(4.10) \quad \mathcal{I}^{(1)} = \langle \theta^a, d\theta^a \rangle = \varpi^*(\mathcal{I}) + \mathcal{J}$$

is called the (first) *prolongation* of (M, \mathcal{I}) , and we are back where we started at the beginning of Section 4. We can construct $M^{(2)} \subset \operatorname{Gr}_n(TM^{(1)})$, and repeat the entire process for $E \in M^{(2)}$ over $e \in M^{(1)}$ that was used for $e \in M^{(1)}$ over $p \in M$. Lemma 4.7 essentially says that $A^{(1)}$ is the tableau bundle $TM^{(2)} \subset T\operatorname{Gr}_n(TM^{(1)})$. Thus, we can construct $M^{(3)}$ over $M^{(2)}$ and re-apply Lemma 4.7, and so on. By the definition of $M^{(1)}$ and (4.10), we have

COROLLARY 4.11. *Every ordinary integral manifold N of $(M^{(1)}, \mathcal{I}^{(1)})$ is also an ordinary integral manifold of (M, \mathcal{I}) . However, the converse might fail, as the smooth connected locus of $M^{(1)}$ may be a strict subset of $\operatorname{Var}_n(\mathcal{I})$.*

Overall, we achieve exact sequences that summarize the entire situation of the tangent spaces of an immersed ordinary integral manifold N of \mathcal{I} , $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)}$, $\mathcal{I}^{(3)}$, \dots

$$(4.12) \quad \begin{aligned} 0 &\rightarrow A \rightarrow W \otimes \wedge^1 V^* \rightarrow H^1(A) \rightarrow 0, \\ 0 &\rightarrow A^{(1)} \rightarrow A \otimes V^* \xrightarrow{\delta} W \otimes \wedge^2 V^* \rightarrow H^2(A) \rightarrow 0, \\ 0 &\rightarrow A^{(2)} \rightarrow A^{(1)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^3 V^* \rightarrow H^3(A) \rightarrow 0, \\ &\vdots \\ 0 &\rightarrow A^{(n-1)} \rightarrow A^{(n-2)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^n V^* \rightarrow H^n(A) \rightarrow 0. \end{aligned}$$

The cokernels $H^1(A)$, $H^2(A)$, \dots , $H^n(A)$ are the *Spencer cohomology* of the tableau A . Even outside the context of exterior differential systems, they are defined for formal tableaux $A \subset W \otimes V^*$ via the exact sequences (4.12) as

$$(4.13) \quad H^p(A) = (A \otimes (\otimes^{p-1} V^*)) / (W \otimes \wedge^p V^*).$$

Spencer cohomology detects functional obstructions to the solution of the initial-value problem on $M^{(p)}$ in the form of *torsion*; this is explained nicely in [IL03, Section 5.6].

Spencer cohomology was a major focus of the formal study of partial differential equations and Lie pseudogroups in the mid-20th century; most notably, [Spe62, Qui64, SS65, GQS66, Gol67, Gar67, Gui68, GK68, GQS70]. As it happens, many of the major results of that era are easy to re-prove under our regularity assumptions on $M^{(1)}$ and using the endovolutive notation from Section 1, particularly when using the involutivity criteria in Section 5 that were detailed in [Smi15]. We demonstrate this in Parts III and IV.

5. Involutivity of Exterior Differential Systems

DEFINITION 5.1 (Cartan's test). A tableau A is called *involutive* if equality holds in Equation (4.9),

$$s_1 + 2s_2 + \cdots + \ell s_\ell = \dim A^{(1)}$$

DEFINITION 5.2. A tableau A is called *formally integrable* if $H^p(A) = 0$ for all $p \geq 2$.

Cartan's test comes from the following consequence of the Cartan–Kähler theorem¹⁵

THEOREM 5.3. *Suppose that (M, \mathcal{I}) is an analytic exterior differential system, that $M^{(1)}$ is a smooth sub-bundle, and that the tableau bundle A has constant Cartan characters over $M^{(1)}$. If A is involutive and formally integrable, then through any point in M , there is an analytic ordinary integral manifold $\iota: N \rightarrow M$. Moreover, such N are parametrized locally by r constants, s_1 functions of 1 variable, s_2 functions of 2 variables, \dots , s_ℓ functions of ℓ variables.*

Somewhat confusingly, the situation in Theorem 5.3 is called *involutivity* of (M, \mathcal{I}) ; that is, an EDS might fail to be involutive even if its tableau is involutive, because there may be nonzero *torsion* in $H^p(A)$, meaning that \mathcal{I} fails to be formally integrable. This means essentially that the ideal \mathcal{I} is being studied on the wrong manifold.

For a beautiful interpretation of Cartan's test that is relevant to the later Sections of this course, read the introduction of [Yan87]. In summary, ordinary integral manifolds are constructed by decomposing the Cauchy problem into a sequence of steps, each of which is determined and has solutions using the Cauchy–Kowalevski theorem.

For fixed spaces W and V^* , involutivity is a closed algebraic condition on tableaux in $W \otimes V^*$. Because the conditions come from Cartan's test, which involves $W \otimes \wedge^2 V^*$, it is not surprising that the conditions are quadratic; however, writing down the precise ideal is a lengthy argument. Doing so was suggested in [BCG+90, Chapter IV§5] and accomplished for general tableaux in [Smi15] following the outline in [Yan87].

THEOREM 5.4 (Involutivity Criteria). *Suppose a tableau is given in generic bases as in (1.14). The tableau is involutive if and only if there exists a basis of W such that*

- (i) B_i^λ is endovolutive in that basis, and
- (ii) $\left(B_i^\lambda B_k^\mu - B_k^\lambda B_i^\mu \right)_b^a = 0$ for all $\lambda < l < k$ and $\lambda \leq \mu < k$ and all $a > s_l$.

This theorem is our main computational tool in Part III.

¹⁵See [BCG+90, Chapter III] or [IL03] for more background on the Cartan–Kähler theorem; it is not our focus here.

5(a). Moduli of Involutive Tableaux. While it seems like a trivial (if lengthy) computation, consider carefully the meaning of Theorem 5.4: We can fix r , n , and Cartan characters s_1, \dots, s_n and then write down an explicit ideal in coordinates whose variety is all of the involutive tableaux with those characters. Hence, we can use computer algebra systems such as Macaulay2, Magma, and Sage to decompose and analyze that ideal using Gröbner basis techniques. With enough computer memory, we can answer the question “What is the moduli of involutive tableaux?” By virtue of Theorem 5.3, this is fairly close to answering the question “What is the moduli of involutive PDEs?”

For example, fix $r = n = 3$ and $(s_1, s_2, s_3) = (3, 2, 0)$. For some coefficients x_0, \dots, x_{15} in the ring S , an endovolutive tableau must be of the form

$$(5.5) \quad (\pi_i^a) = \begin{pmatrix} \alpha_0 & \alpha_3 & x_3\alpha_0 + x_6\alpha_1 + x_9\alpha_2 + x_{12}\alpha_3 + x_{14}\alpha_4 \\ \alpha_1 & \alpha_4 & x_4\alpha_0 + x_7\alpha_1 + x_{10}\alpha_2 + x_{13}\alpha_3 + x_{15}\alpha_4 \\ \alpha_2 & x_0\alpha_0 + x_1\alpha_1 + x_2\alpha_2 & x_5\alpha_0 + x_8\alpha_1 + x_{11}\alpha_2 \end{pmatrix},$$

or in block form like (1.20),

$$(5.6) \quad (B_i^\lambda) = \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_0 & x_1 & x_2 \end{pmatrix} & \begin{pmatrix} x_3 & x_6 & x_9 \\ x_4 & x_7 & x_{10} \\ x_5 & x_8 & x_{11} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} x_{12} & x_{14} & 0 \\ x_{13} & x_{15} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{bmatrix}$$

Involutivity is an affine quadratic ideal \mathcal{G} on $\mathbb{C}(x_0, \dots, x_{15})$ generated by the last rows of $B_2^1 B_3^1 - B_3^1 B_2^1$ and $B_2^1 B_3^2 - B_3^1 B_2^2$, so:

$$(5.7) \quad \mathcal{G} = \begin{cases} x_0x_3 + x_1x_4 + x_2x_5 - x_0x_{11}, \\ x_0x_6 + x_1x_7 + x_2x_8 - x_1x_{11}, \\ x_0x_9 + x_1x_{10}, \\ x_0x_{12} + x_1x_{13} - x_5, \\ x_0x_{14} + x_1x_{15} - x_8. \end{cases}$$

The complete primary decomposition of this ideal reveals two components. The maximal component has dimension 12, and it is described by the fairly boring prime ideal $\{x_0, x_1, x_5, x_8\}$. The other component has dimension 11 and its prime ideal is generated by 27 polynomials. See <http://goo.gl/jGTnMU> for how to compute this in SageMathCell.

Many of your favorite involutive second-order scalar PDEs in three independent variables live somewhere in this variety; see (1.15) and Section 6(c). Up to some notion of equivalence, this is essentially the moduli space of such equations. As seen in Part III, their characteristic varieties are obtained by combining \mathcal{G} with the rank-one ideal \mathcal{R} on $\mathbb{C}[x_0, \dots, x_{15}, a_0, \dots, a_4]$.

However, there is still some ambiguity to be resolved, as it may be that a given abstract tableau admits several endovolutive bases with apparently distinct coordinate descriptions.

5(b). Cauchy retractions. Before proceeding to Part III, it is worthwhile to mention Cauchy retractions, which are much simpler than—and quite distinct from—elements

of the characteristic variety. To confuse matters, many references call these “Cauchy characteristics.” For any differentially closed ideal $\mathcal{I} \subset \Omega^\bullet M$, the Cauchy retractions are the vectors that preserve \mathcal{I} ; that is, $\mathfrak{g} = \{v \in TM : v \lrcorner \mathcal{I} \subset \mathcal{I}\}$. Because \mathcal{I} is differentially closed, the annihilator bundle $\mathfrak{g}^\perp \subset T^*M$ is the smallest Frobenius ideal in $\Omega^\bullet(M)$ that contains \mathcal{I} . Then, for any integral manifold $\iota : N \rightarrow M$, the subspaces $\mathfrak{g} \cap \iota^{(1)}(N)$ form an integrable distribution; that is, \mathfrak{g}_N^\perp is Frobenius as well [Gar67].

Because \mathfrak{g}^\perp is a Frobenius system—a system of ODEs—it is common to redefine (M, \mathcal{I}) so that it is free of Cauchy retractions before proceeding to study its integral manifolds. The separation between \mathfrak{g}^\perp and the characteristic variety Ξ is explored further in [Smi14].

Part III

Characteristic and Rank-One Varieties

Thank you for taking the time to read the enormous amount of background in Parts **I** and **II**. We are ready to define and deconstruct a fascinating mathematical object that lies at the heart of PDE theory.

Here we stand: We have an exterior differential system \mathcal{I} on M . Perhaps this EDS arose from a system of PDEs on M and is equipped with an independence condition ω . The EDS yields a smooth Kähler-regular subbundle $M^{(1)} \subset \text{Gr}_n(TM)$, where any $e \in M^{(1)}$ is an integral element of the original EDS. As a manifold in its own right, $M^{(1)}$ is equipped with tautological bundles V , V^* , W , and A from (4.3). Moreover, A is a subbundle of $W \otimes V^*$, so it is a tableau bundle. Its symbol σ gives a short-exact sequence of bundles:

$$(5.8) \quad 0 \rightarrow A \rightarrow W \otimes V^* \xrightarrow{\sigma} H^1(A) \rightarrow 0.$$

An integral manifold is an immersion $\iota : N \rightarrow M$ such that $\iota_*(T_p N) \in M_{i(p)}^{(1)}$ for all $p \in N$. Let $\iota^{(1)} : N \rightarrow M^{(1)}$ denote the map $p \mapsto e = \iota_*(T_p N)$.

As you read this part, compare it to [IL03, Section 4.6] and [BCG⁺90, Chapter V]. The reader will note that we do not assume that \mathcal{I} is a linear Pfaffian system, nor do we build a prolonged EDS $\mathcal{I}^{(1)}$ using the contact system. Instead we are working with the tautological bundles per Remark 3.13.

6. The Characteristic Variety

The original motivation for the characteristic variety is to see where the initial-value problem becomes ambiguous. That is, given an initial condition for our PDE on a local submanifold of dimension $n-1$, when would the n -dimensional solutions for that initial condition fail to be unique? We express this condition in terms of integral elements.

6(a). via Polar Extension. For an integral element $e' \in \text{Var}_{n-1}(TM)$, we consider its space¹⁶ of integral extensions, called the *polar space*,

$$(6.1) \quad H(e') = \{v : e = e' + \langle v \rangle \in \text{Var}_n(\mathcal{I})\} \subset TM$$

and the *polar equations* comprise its annihilator,

$$(6.2) \quad H^\perp(e') = \{e' \lrcorner \varphi : \varphi \in \mathcal{I}_n\} \subset T^*M.$$

The *polar rank* is $r(e') = \dim H(e') - \dim e' - 1$. If $r(e') = -1$, then e' admits no extensions. If $r(e') = 0$, then e' admits a unique extension to some $e \in \text{Var}_n(\mathcal{I})$.

The case of interest is $r(e') > 0$, meaning that e' admits many extensions, so the initial-value problem from e' to $e = e' + \langle v \rangle$ is ambiguous. For any $e \in M^{(1)}$, we can identify a hyperplane $e' \in \text{Gr}_{n-1}(e)$ with $\xi \in \mathbb{P}e^*$ via $e' = \ker \xi$. Because $e \in M^{(1)} \subset \text{Gr}_n(TM)$ where n is the maximal dimension of integral elements of \mathcal{I} , the function r cannot be positive on an open set of $\mathbb{P}e^*$, so the case $r(e') > 0$ is a closed condition. Moreover, the function $r : \mathbb{P}e^* \rightarrow \mathbb{N}$ is the rank of a linear system of equations, so it defines a

¹⁶The polar space is a vector space thanks to the assumption that \mathcal{I}_n is a finitely-generated $C^\infty(M)$ -module, because that assumption implies that the polar equations over $p \in M$ are a linear subspace of T_p^*M .

Zariski-closed projective algebraic variety. We choose to study that algebraic variety over \mathbb{C} . Hence, the typical definition of the *characteristic variety* of e is

$$(6.3) \quad \Xi_e = \{\xi \in \mathbb{P}e^* \otimes \mathbb{C} : r(\xi^\perp) > 0\} \subset V_e^*.$$

This initial definition is refined in Section 6(b) to produce a scheme. To study properly the ambiguity of the initial-value problem, we want to assign a multiplicity to each $\xi \in \Xi_e$ and decompose Ξ into irreducible components based on the structure of the space $H(\xi^\perp)$.

6(b). via Rank-1 Incidence. Section 2(b) provides another interpretation of the initial-value problem that is much more convenient than (6.3). To see thus, we must tie the polar space $H(e')$ to the tableau A_e of an extension e of e' .

Fix $e \in M^{(1)}$, and suppose that both e and \tilde{e} are integral extensions of $e' = \ker \xi$ for some $\xi \in V^*$. By the definition of $H(e')$, it must be that \tilde{e} lies in $\text{Var}_n(\mathcal{I})$, but we do not know that \tilde{e} lies near e in $M^{(1)}$. Because $\tilde{e} \in \text{Pol}_1(e)$ and $\xi \in \Xi_e$, Lemma 2.10 yields a particular rank-1 projective homomorphism $w \otimes \xi \in W \otimes V^*$ representing \tilde{e} . Because $H(e')$ is a vector space, Lemma 2.16 implies that this $w \otimes \xi$ lies in the tableau A_e . Therefore, because $M^{(1)}$ is an open, connected manifold, we know $w \otimes \xi \in A_e$ is represented by some e_τ that lies in $M^{(1)}$ near e , as in (2.13). That is, any polar pair \tilde{e} of e is detected by A_e and is represented via \arctan_e by a curve of polar pairs in $M^{(1)}$.

On the other hand, for fixed e and ξ , there are various distinct \tilde{e} corresponding to linearly independent w . With Figure 6 in mind, it is easy to see that

$$(6.4) \quad \dim \mathbb{P}\{w \in T_p M/e : w \otimes \xi \in A_e\} = r(\xi^\perp).$$

Recall the rank-one ideal \mathcal{R} from Section 1. Here it applies to vector bundles. As a set, the rank-one subvariety of the tableau is

$$(6.5) \quad \mathcal{E} = A \cap \text{Var } \mathcal{R} = A \cap \{w \otimes \xi : w \in W, \xi \in V^*\}.$$

As a set, the *characteristic variety* Ξ is the projection of \mathcal{E} to V^* . More precisely, Ξ is the *scheme*¹⁷ defined by the *characteristic ideal* \mathcal{M} on V^* that is obtained from the rank-one ideal \mathcal{R} on $A \subset W \otimes V^*$ in the following way: For any $\xi \in V^*$, define $\sigma_\xi : W \rightarrow H^1$ by $\sigma_\xi(w) = \sigma(w \otimes \xi)$. Note that $\dim \ker \sigma_\xi = r(\xi^\perp)$ by (6.4) and (6.5), but this does not account for multiplicity within \mathcal{E} itself. Then the scheme \mathcal{E} is the incidence correspondence¹⁸ of Ξ for the symbol map σ_ξ . See Figure 12.

This interpretation is *amazing*. Suddenly, two completely elementary ideas from Section 1—tableaux of matrices and rank-one matrices—come together to give a concise description of the most subtle structure in PDE theory.

However, the scheme components and multiplicities are still not obvious from Figure 12; they must be obtained by examining the degree of the equations defining $\ker \sigma_\xi$. The powerful third interpretation in Section 7 provides this detail. But first an example.

¹⁷We must study Ξ along with its various components and multiplicities, so it is better to think of it as a scheme than as a simple-minded variety.

¹⁸For more background on the utility of incidence correspondences in algebraic geometry, I recommend the 2013 Columbia Eilenberg lecture series by Joe Harris, [Har13], which I had the pleasure of attending and discussing with Ian Morrison during my time at Fordham University. It is all available on YouTube.

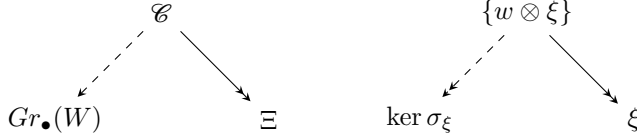


FIGURE 12. The rank-one variety \mathcal{E} is the incidence correspondence for the characteristic variety Ξ , but the scheme multiplicities in Ξ should be obtained as in (7.14).

6(c). Example: The Wave Equation. Consider the PDE $f_{11} + f_{22} = f_{33}$. To do this, we consider the manifold $M = \mathbb{R}^{3+1+3+5} = \mathbb{J}^2(\mathbb{R}^3, \mathbb{R})$ with coordinates $x^1, x^2, x^3, f, p_1, p_2, p_3, p_{11}, p_{12}, q_{13}, q_{22}, q_{23}$. Consider the exterior differential system generated by

$$\begin{aligned}
 \theta^0 &= du - p_1 dx^1 - p_2 dx^2 - p_3 dx^3, \\
 \theta^1 &= dp_1 - p_{11} dx^1 - p_{12} dx^2 - p_{13} dx^3, \\
 \theta^2 &= dp_2 - p_{12} dx^1 - p_{22} dx^2 - p_{23} dx^3, \\
 \theta^3 &= dp_3 - p_{13} dx^1 - p_{23} dx^2 - (p_{11} + p_{22}) dx^3
 \end{aligned}
 \tag{6.6}$$

The derivatives are

$$d \begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \pi_1^1 & \pi_2^1 & \pi_3^1 \\ \pi_1^2 & \pi_2^2 & \pi_3^2 \\ \pi_1^3 & \pi_2^3 & \pi_3^3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \mod \{\theta^0, \theta^1, \theta^2, \theta^3\}
 \tag{6.7}$$

where $\pi_2^1 = \pi_1^2$, $\pi_3^1 = \pi_1^3$, $\pi_3^2 = \pi_2^3$, and $\pi_3^3 = \pi_1^1 + \pi_2^2$.

Changing bases, this tableau is equivalent to an endovolutive one of the form

$$(\pi_i^a) = \begin{pmatrix} \alpha_0 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_4 & \alpha_2 + \alpha_3 \\ \alpha_2 & \alpha_0 & \alpha_1 \end{pmatrix}
 \tag{6.8}$$

Or in block form

$$(B_i^\lambda) = \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{bmatrix}
 \tag{6.9}$$

Note that the third row of both $B_2^1 B_3^1 - B_3^1 B_2^1$ and $B_2^1 B_3^2 - B_3^1 B_2^2$ are zero, so the tableau is involutive by Theorem 5.4.

The rank-one condition is

$$\begin{aligned}
 & \alpha_0 \alpha_4 - \alpha_1 \alpha_3 \\
 & \alpha_0 \alpha_0 - \alpha_2 \alpha_3 \\
 & \alpha_0 \alpha_1 - \alpha_2 \alpha_4 \\
 (6.10) \quad & \alpha_1 \alpha_1 - \alpha_2 \alpha_2 - \alpha_2 \alpha_3 \\
 & \alpha_3 \alpha_1 - \alpha_0 \alpha_4 \\
 & \alpha_3 \alpha_2 + \alpha_3 \alpha_3 - \alpha_4 \alpha_4 \\
 & \alpha_4 \alpha_1 - \alpha_0 \alpha_2 - \alpha_0 \alpha_3
 \end{aligned}$$

After a simple change of basis, this becomes the example (1.2) – (1.4), seen throughout the earlier sections.

7. Guillemin Normal Form and Eigenvalues

In this section, we reinterpret \mathcal{C} and Ξ as properties of the endomorphisms B_i^λ . This section is the key to all of the more advanced results that follow. Our main computation tool is the structure of an endovolutive tableau discussed in Section 1(c), where W and V and A are now the projective bundles over $M^{(1)}$.

The incidence correspondence of Figure 12 is rephrased in Lemma 7.1.

LEMMA 7.1. *If $\xi \in \Xi$, $v \in V$, and $w \in \ker \sigma_\xi \subset W$, then*

$$(7.2) \quad B(\xi)(v)w = \xi(v)w.$$

In particular, w is an eigenvector of $B(\xi)(v)$ for all v .

PROOF. Fix generic bases (u^i) and (z_a) and (u_i) , so that $\xi = \xi_i u^i$ and $w = w^a z_a$ and $v = v^i u_i$. Set $\pi = w \otimes \xi \in \mathcal{C} \subset A$, so $\pi_i^a = w^a \xi_i$ for all a, i , and this π must satisfy the symbol relations (1.10). In particular, $w^a \xi_i = B_{i,b}^{a,\lambda} w^b \xi_\lambda$ for $a > s_i$. Therefore

$$\begin{aligned}
 B(\xi)(v)w &= \sum_{a \leq s_i} \xi_i v^i w^a z_a + \sum_{a > s_i} B_{i,b}^{a,\lambda} w^b \xi_\lambda v^i z_a \\
 (7.3) \quad &= \sum_{a \leq s_i} \xi_i v^i w^a z_a + \sum_{a > s_i} \xi_i v^i w^a z_a \\
 &= \sum_{a,i} \xi_i v^i w^a z_a = \xi(v)w.
 \end{aligned}$$

(Here we see the utility of including the first summand in Equation (1.14).) □

Lemma 7.4 provides a sort of converse.

LEMMA 7.4. *Suppose that A is an endovolutive tableau. Fix $\varphi \in Y^\perp \cong Y^\perp$ and suppose that $w \in \mathbf{W}^-(\varphi)$ such that w is an eigenvector of $B(\varphi)(v)$ for every $v \in V$. Then there is a $\xi \in \Xi$ over $\varphi \in Y^\perp$ such that $w \in \mathbf{W}^1(\varphi)$, so $w \otimes \xi \in A$.*

PROOF. For each $v \in V$, let $\xi(v)$ denote the eigenvalue corresponding to v , so that $\xi(v)w = B(\varphi)(v)w$. Because $B(\varphi)(v)w$ is linear in v , so is $\xi(v)$. Then $\xi = \xi_i u^i \in V^*$.

Therefore, $B(\varphi)(\cdot)w = w \otimes \xi$. In particular, the rank-one condition implies that

$$(7.5) \quad \sum_{\lambda \leq \mu} \varphi_\lambda B_\mu^\lambda w = \xi_\mu w = \sum_{\lambda \leq \mu} \xi_\lambda B_\mu^\lambda w, \quad \forall \mu \leq \ell.$$

This is the same expression as in (1.22), so by comparing recursively over $\mu = 1, 2, \dots, \ell$, we see that $\xi_\lambda = \varphi_\lambda$ for all λ , so $w \in \mathbf{W}^1(\varphi) \subset \mathbf{W}^-(\varphi)$. \square

Lemma 7.4 deserves a warning: There may be multiple ξ over the same φ , for perhaps there are different eigenvectors $w \in \mathbf{W}^-(\varphi)$ admitting different sequences of eigenvalues ξ_ρ , for $\rho > \ell$, associated to the same φ . Moreover, it is not (yet) clear that a mutual eigenvector w exists for every such φ .

But overall it is clear that there is a some relationship between the eigenvalues of B_i^λ and the characteristic variety of an endovolutive tableau A . This relationship is made precise for involutive tableaux using a result from [Gui68].

THEOREM 7.6 (Guillemin normal form). *Suppose that A is involutive. For every $\varphi \in Y^\perp$ and $v \in V$, the restricted homomorphism $B(\varphi)(v)|_{\mathbf{W}^1(\varphi)}$ is an endomorphism of $\mathbf{W}^1(\varphi)$. Moreover, for all $v, \tilde{v} \in V$,*

$$(7.7) \quad [B(\varphi)(v), B(\varphi)(\tilde{v})] \Big|_{\mathbf{W}^1(\varphi)} = 0.$$

Compare Theorem 7.6 to Lemma 4.1 in [Gui68] and Proposition 6.3 in Chapter VIII of [BCG⁺90]. Theorem 7.6 is known as *Guillemin normal form* because it implies that the family of homomorphisms $B(\varphi)(\cdot)$ can be placed in simultaneous Jordan normal form on $\mathbf{W}^1(\varphi)$. It is the “normal form” alluded to in Section 1(b). We defer the proof to Section 9 so we may first see its important consequences.

COROLLARY 7.8. *If A is involutive, then for each $\varphi \in Y^\perp$, there exists some w satisfying the hypotheses of Lemma 7.4. That is, the projection map $\Xi \rightarrow Y^\perp$ is onto. In particular, if A is nontrivial and involutive, then Ξ is nonempty.*

PROOF. Because we are working over \mathbb{C} , the commutativity condition (7.7) guarantees that common eigenvectors exist for the commutative algebra $\{B(\varphi)(v) : v \in V\}$. \square

LEMMA 7.9. *Suppose that A is an involutive tableau. Then the map of projective varieties induced by $\Xi \rightarrow Y^\perp$ is a finite branched cover. In particular, we have the affine dimensions $\dim \Xi = \dim Y^\perp = \ell$.*

PROOF. Fix $\varphi \in Y^\perp$. The set of ξ over φ is nonempty by Corollary 7.8. If it were true that the set of ξ projecting to a particular φ were infinite, then the parameter ξ_i would take infinitely many values in some expression of the form

$$(7.10) \quad \det \left(\sum_{\lambda} \varphi_\lambda B_i^\lambda - \xi_i I \right) = 0.$$

But, the matrix $\sum_{\lambda} \varphi_\lambda B_i^\lambda \in \text{End}(\mathbf{W}_1^-)$ can have at most s_1 eigenvalues. \square

Here we arrive at an easy¹⁹ proof of the main theorem regarding the structure of Ξ .

THEOREM 7.11. *If A is involutive, then $\dim \Xi = \ell - 1$ and $\deg \Xi = s_\ell$.*

PROOF. We work in endovolutive coordinates. From Lemma 7.9, we already know that $\dim \Xi = \ell - 1$.

Fix a generic point $\xi \in \Xi$ over $\varphi \in Y^\perp$. Let $\mathcal{E}_\xi = (\ker \sigma_\xi) \otimes \xi$ denote the fiber over ξ in \mathcal{E} . To understand the scheme Ξ , we must determine the degree of the condition defining \mathcal{E}_ξ . Note that \mathcal{E}_ξ must be a subvariety of $\mathbf{W}^1(\varphi) \otimes \xi$, and $\mathbf{W}^1(\varphi)$ is a linear subspace of W , so the degree of Ξ is the degree of some condition on $\mathbf{W}^1(\varphi)$.

By Lemma 7.1 and (6.5), the condition that \mathcal{E}_ξ is nontrivial is precisely the condition that

$$(7.12) \quad \det \left(\sum_{\lambda} \xi_{\lambda} B_i^{\lambda} - \xi_i I \right) = 0, \quad \forall i.$$

Since we may restrict our attention to $\mathbf{W}^1(\varphi) \otimes \xi$, the condition (7.12) for $i \leq \ell$ is automatic by (1.23). Hence, only these terms contribute to the non-linear part of the ideal:

$$(7.13) \quad \det \left(\sum_{\lambda} \xi_{\lambda} B_{\varrho}^{\lambda} - \xi_{\varrho} I \right) = 0, \quad \forall \varrho > \ell.$$

or, without coordinates, the defining equations of \mathcal{E}_ξ are

$$(7.14) \quad \det (B(\xi)(v) - \xi(v)I) = 0, \quad \forall v \in Y.$$

For a particular v , this is the characteristic polynomial of $B(\xi)(v)$ as an endomorphism of $\mathbf{W}^1(\varphi)$. By involutivity and Theorem 7.6, all $B(\xi)(v)$ for $v \in Y$ admit the same Jordan-block form, so they admit the same factorization type for their respective characteristic polynomials. That means it suffices to consider a single v . By definition, the characteristic polynomial of $B(\xi)(v)|_{\mathbf{W}^1(\varphi)}$ has degree $\dim \mathbf{W}^1(\varphi)$ at generic φ . Therefore, $\deg \Xi = s_\ell$ follows from Lemma 1.25. \square

Theorems 7.6 and 7.11 provide a powerful interpretation of the form of an involutive tableau seen in Theorem 5.4 and Figure 3; the first ℓ columns represent a projection of Ξ , as in Lemma 7.9, and the rank-one incidence correspondence in Figure 12 is precisely the eigenvector condition on the appropriate subspaces. It is peculiar and interesting that these results were discovered in the opposite order historically, as explored in Section 9.

Moreover, the proof of Theorem 7.11—in particular Equation (7.14)—gives a precise understanding of Ξ as a scheme: The characteristic scheme (in the sense of PDE) is merely a scheme of characteristic equations (in the sense of linear algebra)! The components of Ξ correspond to the various Jordan blocks apparent in (7.14), and the multiplicity of each component is the dimension of that generalized eigenspace. The sheets of the finite

¹⁹It is easy in the sense that we have the explicit polynomials of \mathcal{M} in hand, and they are recognizable as the familiar eigenvector equations. The reader should compare (7.14) to the descriptions provided in [BCG⁺90] and [IL03]. Both references defer their decomposition of Ξ to the abstract Grothendieck–Riemann–Roch theorem. Hence, neither reference indicates how to compute the scheme by hand for general tableaux. While details are given in [BCG⁺90] in the simple case of rectangular tableaux, a complete description is achieved here because of the normal form provided by Theorem 5.4.

branched cover $\Xi \rightarrow Y^\perp$ come from different generalized eigenspaces where the first ℓ eigenvalues match. See Section 8 for how to compute this.

8. Examples

8(a). Dimension-0 Examples. Consider some cases of involutive tableaux with $(s_1, s_2, s_3) = (4, 0, 0)$.

$$(8.1) \quad (\pi_i^a) = \begin{pmatrix} \pi_1^1 & \pi_2^1 & \pi_3^1 \\ \pi_1^2 & \pi_2^2 & \pi_3^2 \\ \pi_1^3 & \pi_2^3 & \pi_3^3 \\ \pi_1^4 & \pi_2^4 & \pi_3^4 \end{pmatrix}.$$

Or, in endovolution block form:

$$(8.2) \quad (B_i^\lambda) = [I_4 \quad B_2^1 \quad B_3^1].$$

The characteristic ideal \mathcal{M} will have degree $s_\ell = 4$ and projective dimension $\ell - 1 = 0$. That is, Ξ will be 4 points, counted with multiplicity. The involutivity condition is $0 = B_2^1 B_3^1 - B_3^1 B_2^1$ (all rows); that is, the matrices commute. Thus the matrices B_2^1 and B_3^1 must have compatible Jordan-block forms; they span a commutative algebra. In these examples, we will use colors to emphasize the distinct generalized eigenspaces.

One possibility is that the matrices are diagonal with distinct Jordan blocks:

$$(8.3) \quad A = \left\{ \begin{pmatrix} \alpha_1 & c_1 \alpha_1 & d_1 \alpha_1 \\ \alpha_2 & c_2 \alpha_2 & d_2 \alpha_2 \\ \alpha_3 & c_3 \alpha_3 & d_3 \alpha_3 \\ \alpha_4 & c_4 \alpha_4 & d_4 \alpha_4 \end{pmatrix} : \alpha_a \in \mathbb{C} \right\}$$

In this case, the rank-1 cone is

$$(8.4) \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \otimes [1 : c_1 : d_1], \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \otimes [1 : c_2 : d_2], \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes [1 : c_3 : d_3], \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes [1 : c_4 : d_4] \right\}$$

Each point $\xi \in \Xi$ has multiplicity 1.

Another possibility is that they are diagonal, but an eigenspace has dimension 2.

$$(8.5) \quad A = \left\{ \begin{pmatrix} \alpha_1 & c_1 \alpha_1 & d_1 \alpha_1 \\ \alpha_2 & c_1 \alpha_2 & d_1 \alpha_2 \\ \alpha_3 & c_3 \alpha_3 & d_3 \alpha_3 \\ \alpha_4 & c_4 \alpha_4 & d_4 \alpha_4 \end{pmatrix} : \alpha_a \in \mathbb{C} \right\}$$

In this case, the rank-1 cone is

$$(8.6) \quad \mathcal{C} = \left\{ \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \otimes [1 : c_1 : d_1], \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes [1 : c_3 : d_3], \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes [1 : c_4 : d_4] \right\}.$$

One point $\xi \in \Xi$ has multiplicity 2; in particular, the fiber $\ker \sigma_\xi$ for $\xi = [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1]$ should be seen as a \mathbb{P}^1 . This is reflected clearly in (7.14), because $\xi = [\xi_1 : \xi_2 : \xi_3] = [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1]$ is a root of degree 2 for any v :

$$\begin{aligned}
 (8.7) \quad 0 &= \det (\xi_1(v^2 B_2^1 + v^3 B_3^1) - (\xi_2 v^2 + \xi_3 v_3)I) \\
 &= \left| v^2 \begin{pmatrix} \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 0 & 0 & 0 \\ 0 & \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 0 & 0 \\ 0 & 0 & \textcolor{blue}{c}_3 - \textcolor{red}{c}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{c}_4 - \textcolor{red}{c}_1 \end{pmatrix} + v^3 \begin{pmatrix} \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 0 & 0 & 0 \\ 0 & \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 0 & 0 \\ 0 & 0 & \textcolor{blue}{d}_3 - \textcolor{red}{d}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{d}_4 - \textcolor{red}{d}_1 \end{pmatrix} \right| \\
 &= v^2(\textcolor{red}{c}_1 - \textcolor{red}{c}_1)^2(\textcolor{blue}{c}_3 - \textcolor{red}{c}_1)(\textcolor{green}{c}_4 - \textcolor{red}{c}_1) + v^3(\textcolor{red}{d}_1 - \textcolor{red}{d}_1)^2(\textcolor{blue}{d}_3 - \textcolor{red}{d}_1)(\textcolor{green}{d}_4 - \textcolor{red}{d}_1).
 \end{aligned}$$

Another possibility is that there is a 2×2 block:

$$(8.8) \quad A = \left\{ \begin{pmatrix} \alpha_1 & \textcolor{red}{c}_1 \alpha_1 + \alpha_2 & \textcolor{red}{d}_1 \alpha_1 + \alpha_2 \\ \alpha_2 & \textcolor{red}{c}_1 \alpha_2 & \textcolor{red}{d}_1 \alpha_2 \\ \alpha_3 & \textcolor{blue}{c}_3 \alpha_3 & \textcolor{blue}{d}_3 \alpha_3 \\ \alpha_4 & \textcolor{green}{c}_4 \alpha_4 & \textcolor{green}{d}_4 \alpha_4 \end{pmatrix} : \alpha_a \in \mathbb{C} \right\}$$

In this case, the rank-1 cone is

$$(8.9) \quad \mathcal{C} = \left\{ \begin{bmatrix} \textcolor{red}{1} \\ \textcolor{red}{0} \\ 0 \\ 0 \end{bmatrix} \otimes [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1], \begin{bmatrix} 0 \\ 0 \\ \textcolor{blue}{1} \\ 0 \end{bmatrix} \otimes [1 : \textcolor{blue}{c}_3 : \textcolor{blue}{d}_3], \begin{bmatrix} 0 \\ 0 \\ 0 \\ \textcolor{green}{1} \end{bmatrix} \otimes [1 : \textcolor{green}{c}_4 : \textcolor{green}{d}_4] \right\}.$$

Note that the fiber over of \mathcal{C} over Ξ has dimension 1 in each case; however, the first point has multiplicity 2. We see that the dimension of the fiber is insufficient to measure the multiplicity of the scheme Ξ , because the incidence correspondence involves the ideal \mathcal{R} . We can see this because of the structure of the rank-1 matrices: the upper 2×2 minors vanish if and only if $\alpha_2 \alpha_2 = 0$, so the fiber $\ker \sigma_\xi$ for $\xi = [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1]$ should be seen as a \mathbb{P}^0 of degree 2. This is reflected clearly in (7.14), because $\xi = [\xi_1 : \xi_2 : \xi_3] = [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1]$ is a root of degree 2 for any v :

$$\begin{aligned}
 (8.10) \quad 0 &= \det (\xi_1(v^2 B_2^1 + v^3 B_3^1) - (\xi_2 v^2 + \xi_3 v_3)I) \\
 &= \left| v^2 \begin{pmatrix} \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 1 & 0 & 0 \\ 0 & \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 0 & 0 \\ 0 & 0 & \textcolor{blue}{c}_3 - \textcolor{red}{c}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{c}_4 - \textcolor{red}{c}_1 \end{pmatrix} + v^3 \begin{pmatrix} \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 1 & 0 & 0 \\ 0 & \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 0 & 0 \\ 0 & 0 & \textcolor{blue}{d}_3 - \textcolor{red}{d}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{d}_4 - \textcolor{red}{d}_1 \end{pmatrix} \right| \\
 &= v^2(\textcolor{red}{c}_1 - \textcolor{red}{c}_1)^2(\textcolor{blue}{c}_3 - \textcolor{red}{c}_1)(\textcolor{green}{c}_4 - \textcolor{red}{c}_1) + v^3(\textcolor{red}{d}_1 - \textcolor{red}{d}_1)^2(\textcolor{blue}{d}_3 - \textcolor{red}{d}_1)(\textcolor{green}{d}_4 - \textcolor{red}{d}_1).
 \end{aligned}$$

Finally, consider the case where both types of multiplicity occur. For example,

$$(8.11) \quad A = \left\{ \begin{pmatrix} \alpha_1 & \textcolor{red}{c}_1 \alpha_1 + \alpha_2 & \textcolor{red}{d}_1 \alpha_1 + \alpha_2 \\ \alpha_2 & \textcolor{red}{c}_1 \alpha_2 & \textcolor{red}{d}_1 \alpha_2 \\ \alpha_3 & \textcolor{red}{c}_1 \alpha_3 & \textcolor{red}{d}_1 \alpha_3 \\ \alpha_4 & \textcolor{green}{c}_4 \alpha_4 & \textcolor{green}{d}_4 \alpha_4 \end{pmatrix} : \alpha_a \in \mathbb{C} \right\}$$

In this case, the rank-1 cone is

$$(8.12) \quad \mathcal{C} = \left\{ \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix} \otimes [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1], \begin{bmatrix} 0 \\ 0 \\ 0 \\ \textcolor{green}{1} \end{bmatrix} \otimes [1 : \textcolor{green}{c}_4 : \textcolor{green}{d}_4] \right\}.$$

The scheme structure of Ξ is apparent here. The point $\xi = [1 : \textcolor{red}{c}_1 : \textcolor{red}{d}_1]$ appears in two components, which correspond to the factorization of

$$(8.13) \quad 0 = \det(\xi_1(v^2 B_2^1 + v^3 B_3^1) - (\xi_2 v^2 + \xi_3 v_3)I) \\ = \left| v^2 \begin{pmatrix} \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 1 & 0 & 0 \\ 0 & \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 0 & 0 \\ 0 & 0 & \textcolor{red}{c}_1 - \textcolor{red}{c}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{c}_4 - \textcolor{red}{c}_1 \end{pmatrix} + v^3 \begin{pmatrix} \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 1 & 0 & 0 \\ 0 & \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 0 & 0 \\ 0 & 0 & \textcolor{red}{d}_1 - \textcolor{red}{d}_1 & 0 \\ 0 & 0 & 0 & \textcolor{green}{d}_4 - \textcolor{red}{d}_1 \end{pmatrix} \right| \\ = v^2(\textcolor{red}{c}_1 - \textcolor{red}{c}_1)^2(\textcolor{red}{c}_1 - \textcolor{red}{c}_1)(\textcolor{green}{c}_4 - \textcolor{red}{c}_1) + v^3(\textcolor{red}{d}_1 - \textcolor{red}{d}_1)^2(\textcolor{red}{d}_1 - \textcolor{red}{d}_1)(\textcolor{green}{d}_4 - \textcolor{red}{d}_1).$$

From the perspective of \mathcal{C} , these components correspond to the rank-1 matrices

$$(8.14) \quad \begin{pmatrix} \alpha_1 & \textcolor{red}{c}_1 \alpha_1 & \textcolor{red}{d}_1 \alpha_1 \\ \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} \\ \alpha_3 & \textcolor{red}{c}_1 \alpha_3 & \textcolor{red}{d}_1 \alpha_3 \\ 0 & 0 & 0 \end{pmatrix}$$

The fiber should be seen as two components, a \mathbb{P}^1 and a \mathbb{P}^0 . Overall, this point has multiplicity 3.

REMARK 8.15. For readers interested in hydrodynamic integrability criteria, take a moment to compute $\text{Sec}_k(\mathcal{C})$ and $\text{Sec}_k(\Xi)$, $k = 2, 3$, in each of these cases. Note that hyperbolic systems of conservation laws have $s_1 = n$ and take the non-degenerate diagonal form of the first example, over \mathbb{R} .

8(b). Dimension-1 Examples. Consider an involutive tableau with $(s_1, s_2, s_3) = (2, 1, 0)$.

$$(8.16) \quad (\pi_i^a) = \begin{pmatrix} \boxed{\pi_1^1} & \boxed{\pi_2^1} & \pi_3^1 \\ \boxed{\pi_1^2} & \pi_2^2 & \pi_3^2 \end{pmatrix}.$$

Or, in endovolutive block form:

$$(8.17) \quad (B_i^\lambda) = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ x_0 & x_1 \end{pmatrix} & \begin{pmatrix} x_2 & x_3 \\ x_4 & x_5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} x_6 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

The characteristic ideal \mathcal{M} will have degree $s_\ell = 1$ and projective dimension $\ell - 1 = 1$. That is, Ξ will be a single curve.

For the sake of concreteness, let us assume that the coefficients are:

$$(8.18) \quad (B_i^\lambda) = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \frac{1}{9} & 0 \end{pmatrix} & \begin{pmatrix} 5 & 0 \\ 1 & 5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix},$$

so that

$$(8.19) \quad (\pi_i^a) = \begin{pmatrix} \alpha_0 & \alpha_2 & 5\alpha_0 + 9\alpha_2 \\ \alpha_1 & \frac{1}{9}\alpha_0 & \alpha_0 + 5\alpha_1 \end{pmatrix}$$

The rank-1 ideal is just $\alpha_0\alpha_0 - 9\alpha_1\alpha_2 = 0$. Write a generic element of \mathcal{C} as $[\alpha_0 : \alpha_1 : \alpha_2] = [3\tau : 1 : \tau^2]$, like so:

$$(8.20) \quad \begin{pmatrix} 3\tau & \tau^2 & 15\tau + 9\tau^2 \\ 1 & \frac{1}{3}\tau & 5 + 3\tau \end{pmatrix}.$$

Thus, a generic element of ξ is of the form $\xi = [3 : \tau : 15 + 9\tau]$ with fiber $\begin{bmatrix} 3\tau \\ 1 \end{bmatrix}$.

Using (7.14), the characteristic scheme of $\xi = [3 : \xi_2 : \xi_3]$ is generated by $0 = \det(\xi_1 v^3 B_3^1 + \xi_2 v^3 B_3^2 - \xi_3 v^3 I_2)$, restricted to the space $\mathbf{W}^1(\xi) \subset W$, which is 1-dimensional. Write τ for ξ_2 ; so we are trying to find $\xi = [3 : \tau : \xi_3]$ over $\varphi = [3 : \tau : 0]$ as in Lemma 7.4. The space $\mathbf{W}^1(\varphi)$ is the space spanned by $\begin{bmatrix} 3\tau \\ 1 \end{bmatrix}$. Hence, the single linear sheet of the characteristic variety over $[3 : \tau : 0]$ is given by $[3 : \tau : 15 + 9\tau]$.

8(c). Dimension 1 Exercise. Now is the time go back and re-read the example (1.4) and see how it fits into Sections 6(c) and 5(a). The wave-equation example offers a single \mathbb{P}^1 whose fiber is also a \mathbb{P}^1 . By choosing appropriate coefficients, you should be able to produce examples with $(s_1, s_2, s_3) = (3, 2, 0)$ with various other components and multiplicities.

In principle, you can choose any Cartan characters, and choose coefficients subject to Theorem 5.4 to build examples in this way.

9. Results of Guillemin and Quillen

As in the analogy Section 1(b), normal forms often reveal shortcuts to other advanced ideas.

Guillemin's proof of Theorem 7.6 made use of two results derived from Quillen's thesis [Qui64]. In this section, we see how these results become easier using Theorem 5.4. (Note that Theorem 5.4 and Theorem 7.6 are *not* equivalent. Theorem 5.4 is strictly stronger; it is easy to construct endovolutive tableaux that satisfy the conclusion of (7.7) but are not involutive. See [Smi15].)

Recall the Spencer cohomology groups from Section 4(b). For any $\varphi \in V^*$, wedging by φ gives a map $W \otimes \wedge^p V^* \rightarrow W \otimes \wedge^{p+1} V^*$. This induces a map on the quotient spaces, $H^p(A) \rightarrow H^{p+1}(A)$.

THEOREM 9.1 (Quillen's Exactness Theorem). *Suppose A is an involutive tableau, and that $\varphi \notin \Xi_A$. Then the sequence of maps by $\wedge\varphi$,*

$$0 \rightarrow A \rightarrow H^1(A) \rightarrow H^2(A) \rightarrow \cdots \rightarrow H^n(A) \rightarrow 0,$$

is exact.

In [Qui64], this theorem is proven using enormous commutative diagrams. In our context, with Theorem 5.4 in hand, we can prove an easy version of Quillen's result, in the form of Lemma 9.3. Lemma 9.3 is a consequence of 9.2, which for us is an easy corollary of Theorem 5.4. This corollary is called Theorem A in [Gui68], where it was proved using a large diagram chase using Quillen's exactness theorem 9.1.

COROLLARY 9.2 (Quillen, Guillemin). *If A is involutive, then $A|_U$ is involutive, and the natural map between prolongations $A^{(1)} \rightarrow (A|_U)^{(1)}$ is bijective.*

PROOF. The first part is an immediate consequence of Theorem 5.4, as the quadratic condition still holds if the range of indices λ, μ, i, j is truncated at ℓ (or greater). In particular, the generators $(\pi_\lambda^a)_{a \leq s_\lambda}$ of A are preserved.

The second part is similarly immediate, using the proof of Theorem 5.4 given in [Smi15]: the contact relation $\pi_\mu^a = Z_{\mu,i}^a u^i$ for $a \leq s_\lambda$ gives coordinates $Z_{\mu,i}^a$ to the prolongation $A^{(1)} \subset A \otimes V^*$, and the $s_1 + 2s_2 + \dots + \ell s_\ell$ independent generators are precisely those $Z_{\mu,\lambda}^a$ with $a \leq s_\mu$ and $\lambda \leq \mu \leq \ell$. Since they involve no indices $i > \ell$, these generators remain independent when the range of indices is truncated at ℓ . \square

Now we come to our simplified version of Theorem 9.1. Compare Lemma 9.3 to the exact sequence (3.4)₂ in [Gui68].

LEMMA 9.3. *Recall that U^\perp is a complement to $Y^\perp \subset V^*$, so that $V^* = Y^\perp \oplus U^\perp$. For A involutive, the sequence*

$$0 \rightarrow W \otimes S^2 U^\perp \rightarrow H^1 \otimes U^\perp \xrightarrow{\delta} H^2$$

is exact.

PROOF. This proof is just an explicit description of the maps in a basis and an application of Corollary 9.2. Let (u^i) be a basis for V^* such that (u^λ) is a basis for Y^\perp and (u^ϱ) is a basis for U^\perp , using the index convention (1.8) from Section 1.

The sequence makes sense because we can split the Spencer sequence (4.12) as $W \otimes V^* = A \oplus H^1$ by identifying the space H^1 with $\{\sum_{a > s_i} \pi_i^a(z_a \otimes u^i)\} \subset W \otimes V^*$, which is the space spanned by the unshaded entries in Figure 1. Using this identification, two elements $\sum_{a > s_i} \pi_i^a(z_a \otimes u^i)$ and $\sum_{a > s_i} \hat{\pi}_i^a(z_a \otimes u^i)$ of $W \otimes V^*$ are equivalent in H^1 if and only if $\pi_i^a - \hat{\pi}_i^a = \sum_{b \leq s_\lambda} B_{i,b}^{a,\lambda} z_i^b$ for some $\{z_i^a : a \leq s_i\}$, the shaded entries in Figure 1. In other words, the projection $W \otimes V^* \rightarrow H^1$ is defined by (1.10), and the projection $W \otimes V^* \rightarrow A$ is defined by the projection onto the orange generator components in Figure 1, those π_λ^a with $a \leq s_\lambda$.

Since $s_\varrho = 0$ for all $\varrho > \ell$, the inclusion $W \otimes U^\perp \subset W \otimes V^*$ is an inclusion $W \otimes U^\perp \subset H^1$. Hence, the inclusion is understood as

$$(9.4) \quad W \otimes S^2 U^\perp \subset (W \otimes U^\perp) \otimes U^\perp \subset H^1 \otimes U^\perp.$$

An element of $H^1 \otimes U^\perp$ is written in $W \otimes V^* \otimes U^\perp$ as

$$(9.5) \quad P = \sum_{a > s_\lambda} P_{\lambda,\varsigma}^a(z_a \otimes u^\lambda \otimes u^\varsigma) + \sum_{a > 0} P_{\varrho,\varsigma}^a(z_a \otimes u^\varrho \otimes u^\varsigma).$$

The image $\delta(H^1 \otimes U^\perp)$ in H^2 is

$$(9.6) \quad \delta(H^1 \otimes V^*) \subset \delta(W \otimes V^* \otimes V^*) \subset W \otimes \wedge^2 V^*,$$

so $\delta P \in W \otimes \wedge^2 V^*$ is of the form

$$(9.7) \quad \delta P = \sum_{a > s_\lambda} P_{\lambda, \varsigma}^a (z_a \otimes u^\lambda \wedge u^\varsigma) + \sum_{a > 0} \frac{1}{2} (P_{\varrho, \varsigma}^a - P_{\varsigma, \varrho}^a) (z_a \otimes u^\varrho \wedge u^\varsigma).$$

Recall that $H^2 = \frac{W \otimes \wedge^2 V^*}{\delta_\sigma(A \otimes V^*)}$. So, $\delta P \equiv 0$ in H^2 if and only if there is some $T \in A \otimes V^*$ such that $\delta_\sigma(T) = \delta(P)$ in $W \otimes \wedge^2 V^*$. Looking at (9.7), it is apparent that such T must have $\delta_\sigma(T|_U) = 0$, as $\delta(P)$ has no $Y^\perp \wedge Y^\perp$ terms. By involutivity and Corollary 9.2, we consider the involutive tableau

$$(9.8) \quad 0 \rightarrow A|_U \rightarrow W \otimes Y^\perp \xrightarrow{\sigma|_U} H_U^1 \rightarrow 0$$

with prolongation

$$(9.9) \quad 0 \rightarrow (A|_U)^{(1)} \rightarrow A|_U \otimes Y^\perp \xrightarrow{\delta_\sigma|_U} W \otimes \wedge^2 Y^\perp \rightarrow H_U^2 \rightarrow 0.$$

Therefore, $T|_U \in A|_U \otimes Y^\perp$ lies in the kernel of $\delta_\sigma|_U$, so $T|_U \in (A|_U)^{(1)}$. Therefore, Corollary 9.2 tells us $T \in A^{(1)}$. That is, $\delta(P) \equiv 0 \in H^2$ if and only if $\delta(P) = \delta_\sigma(T) = 0$.

Therefore, $\delta(P) \equiv 0 \in H^2$ if and only if $P_{\lambda, \varsigma}^a = 0$ and $P_{\varrho, \varsigma}^a = P_{\varsigma, \varrho}^a$ on these index ranges. This occurs if and only if $P = P_{\varrho, \varsigma}^a (z_a \otimes u^\varrho \otimes u^\varsigma)$, meaning $P \in W \otimes S^2 U^\perp$. \square

We are ready to prove Theorem 7.6. The structure of the proof is identical to the original proof in [Gui68].

PROOF OF THEOREM 7.6. Suppose that $w \in \mathbf{W}^1(\varphi)$, so that $\pi = B(\varphi)(\cdot)w = w \otimes \varphi + J$ for some $J \in W \otimes U^\perp$ with $J_\varrho = J_\varrho^a z_a \in \mathbf{W}^-(\varphi)$ for all ϱ . First, we must show that the span of the columns J_ϱ of J lies in $\mathbf{W}^1(\varphi)$.

Consider the element $-J \otimes \varphi = -J_\varrho^a \varphi_\lambda (z_a \otimes u^\lambda \otimes u^\varrho) \in H^1 \otimes U^\perp$. Because $z \otimes \varphi + J \in A$, it must be that $z \otimes \varphi \otimes \varphi \in W \otimes V^* \otimes V^*$ represents the same point in $H^1 \otimes U^\perp$. So, we can compute

$$(9.10) \quad -J_\varrho^a \varphi_\lambda (z_a \otimes u^\lambda \wedge u^\varrho) \equiv z \otimes \varphi \wedge \varphi = 0 \in H^2$$

By Corollary 9.2, there exists $Q = Q_{\varrho, \varsigma}^a (z_a \otimes u^\varsigma \otimes u^\varrho) \in W \otimes S^2 U^\perp$ such that $-J \otimes \varphi - Q \in A \otimes U^\perp$. That is, writing $Q_\varrho = Q_{\varrho, \varsigma}^a (z_a \otimes u^\varsigma) \in W \otimes Y^\perp$, we have $J_\varrho \otimes \varphi + Q_\varrho \in A$ for all ϱ , meaning $J_\varrho \in \mathbf{W}^1(\varphi)$ for all ϱ . Therefore, for any $v \in V$, we have $B(\varphi)(v)z = \varphi(v)z + J(v) \in \mathbf{W}^1(\varphi)$.

Now, mapping again, $B(\varphi)(\cdot)J_\varrho = J_\varrho \otimes \varphi + Q_\varrho$, so $B(\varphi)(u_\varsigma)J_\varrho = Q_{\varrho, \varsigma}$, which is already known to be symmetric in ϱ, ς . Therefore,

$$(9.11) \quad \begin{aligned} B(\varphi)(\tilde{v}) B(\varphi)(v)z &= B(\varphi)(\tilde{v}) (\varphi(v)z + J(v)) \\ &= \varphi(v) B(\varphi)(\tilde{v})z + u^\varrho(v) B(\varphi)(\tilde{v})J_\varrho \\ &= \varphi(v) (\varphi(\tilde{v})z + J(\tilde{v})) + u^\varrho(v) (\varphi(\tilde{v})J_\varrho + Q_{\varrho}(\tilde{v})) \\ &= \varphi(v)\varphi(\tilde{v})z + \varphi(v)J(\tilde{v}) + \varphi(\tilde{v})J(v) + Q(v, \tilde{v}). \end{aligned}$$

This is symmetric in v, \tilde{v} , giving the commutativity condition (7.7) \square

It is interesting to see the inversion of logic that happened here. In the original literature, the overall implications are

$$9.1 \rightarrow 9.3 \rightarrow 9.2 \rightarrow 7.6.$$

But, the arguments here give the overall implication

$$5.4 \rightarrow 9.2 \rightarrow 9.3 \rightarrow 7.6.$$

However, we can write a shorter proof of Theorem 7.6 that relies Theorem 5.4 more directly, avoiding the general results of Quillen. For motivation, consider the following trivial corollary of Theorem 5.4 that is obtained by setting $\lambda = \mu$:

COROLLARY 9.12. *Under the assumptions of Theorem 5.4, $B(u^\lambda)(v)$ is an endomorphism of $\mathbf{W}^-(u^\lambda)$ such that for all $v, \tilde{v} \in Y$,*

$$[B(u^\lambda)(v), B(u^\lambda)(\tilde{v})] = 0.$$

ALTERNATE PROOF OF THEOREM 7.6. Fix $\varphi \in Y^\perp$, and suppose that $w \in \mathbf{W}^1(\varphi)$. We must verify that all maps $B(\varphi)(v)$ preserve $\mathbf{W}^1(\varphi)$ and that they commute. Note that the definition of $\mathbf{W}^1(\varphi)$ in Equation 1.24 depends on the choice of subspace Y^\perp but not on its basis, so we may verify these conditions using any basis we like.

First a trivial case: if it happens that $\varphi \in \Xi \cap Y^\perp$, then $B(\varphi)(v)w = \varphi(v)w \in \mathbf{W}^1(\varphi)$ is a rescaling, and it is immediate that $[B(\varphi)(v), B(\varphi)(\tilde{v})] = 0$.

Otherwise, we have $\varphi \notin \Xi$. Then we may choose a generic basis of V^* in which $\varphi = u^1$. Moreover, we may use that basis to construct an endovolutive basis of W . By Corollary 9.12, it suffices to prove in this basis that $\mathbf{W}^1(u^1)$ is preserved by every $B(u^1)(v)$. Write $B(\varphi)(\cdot)w = w \otimes u^1 + J$, and examine (1.22) on a column J_ϱ of J . For each $\mu = 1, \dots, \ell$, we must verify

$$(9.13) \quad 0 = (B_\mu^1 - \delta_\mu^1 I) J_\varrho = (B_\mu^1 - \delta_\mu^1 I) B_\varrho^1 w = (B_\mu^1 B_\varrho^1 - \delta_\mu^1 B_\varrho^1) w$$

If $\mu = 1$, then this is immediate, since $B_1^1 = I_{s_1}$.

If $\mu \neq 1$, then we are verifying $0 = (B_\mu^1 B_\varrho^1 - 0)w$. Note that $B_\mu^1 w = 0$, since $B(\varphi)(\cdot)w = w \otimes \varphi + J = w \otimes u^1 + J$. Moreover, by Theorem 5.4, we have

$$(9.14) \quad 0 = (B_\mu^1 B_\varrho^1 - B_\varrho^1 B_\mu^1)_b^a w^b = (B_\mu^1 B_\varrho^1)_b^a w^b$$

for $a > s_\mu$. Therefore, $B_\mu^1 B_\varrho^1$ lies in $\mathbf{W}^-(\mu)$. On the other hand, note that the output of B_μ^1 lies in \mathbf{W}_μ^+ by the construction of the maps B_μ^λ from the reduced symbol in Section 1(c). Combining these, we see that $B_\mu^1 B_\varrho^1 w$ lies in $\mathbf{W}_\mu^- \cap \mathbf{W}_\mu^+ = 0$.

Hence, the space $\mathbf{W}^1(\varphi)$ is preserved by $B(\varphi)(v)$ for all v . By Corollary 9.12, they commute. \square

On the theoretical side, it would be interesting to see how many of the hard classical theorems in the subject can be re-proven with elementary techniques. Specifically, the proof of Lemma 9.3 suggests an elementary proof of Quillen's exactness theorem. The other hard theorem is the integrability of the characteristic variety, and a proof of that theorem using Guillemin's original formulation is the subject of [GQS70]. That result was applied immediately to study primitive Lie pseudogroups.

10. Prolongation

How does the characteristic scheme change under prolongation? The short answer is that it does not! This does not depend on endovolutivity or involutivity.

Recall that $A^{(1)}$ is a tableau within $A \otimes V^*$. An element of $A^{(1)}$ is $P \in A \otimes V^*$. Using any bases for V, W, A , we may write P as $P_{i,j}^a z_a \otimes u^i \otimes u^j$, with the additional condition that $P_{i,j}^a = P_{j,i}^a$ from (4.6). Let $\mathcal{C}^{(1)}$ denote the rank-one elements of $A^{(1)}$, and let $\Xi^{(1)}$ denote its projection to V^* , as in Section 6(b).

THEOREM 10.1. *If $\pi \otimes \xi \in \mathcal{C}^{(1)}$, then $\pi = w \otimes \xi \in \mathcal{C}$ for some $w \in \ker \sigma_\xi$. Conversely, if $w \otimes \xi \in \mathcal{C}$, then $(w \otimes \xi) \otimes \xi \in \mathcal{C}^{(1)}$. In particular, $\Xi \cong \Xi^{(1)}$ as schemes.*

PROOF. Suppose that $\pi \otimes \xi \in \mathcal{C}^{(1)}$ for some $\pi \in A$ and $\xi \in V^*$. That is, $P \in A^{(1)}$ and $P = \pi \otimes \xi$, so $P_{i,j}^a = \pi_i^a \xi_j$, and $\pi_i^a \xi_j = \pi_j^a \xi_i$ for all a, i, j .

Let λ be the minimum index such that $\xi_\lambda \neq 0$. Then $\pi_\lambda^a \xi_i = \pi_i^a \xi_\lambda$, so column i of (π_i^a) is a multiple—namely ξ_i / ξ_λ —of column λ for all i . Therefore, (π_i^a) is rank-one, and there is some w with $\pi = w \otimes \xi$. The converse is immediate. \square

REMARK 10.2. Theorem 10.1 is used sometimes as a method for computing the characteristic variety, as follows: Given a tableau (π_i^a) whose entries might depend on $e \in M^{(1)}$, consider $(\xi_i) \mapsto (\pi_i^a \xi_j - \pi_j^a \xi_i)$ as a map $V^* \rightarrow W \otimes \wedge^2 V^*$; that is, a map from \mathbb{C}^n to $\mathbb{C}^{r \binom{n}{2}}$. For a general point in $\xi \in V^*$, this map has rank at least 1. Its rank falls to 0 if and only if $\xi \in \Xi$. But, I don't recommend this method. If you have (π_i^a) in hand and want to compute 2×2 minors of something, you might as well compute the 2×2 minors of (π_i^a) itself to find \mathcal{C} .

11. Characteristic Sheaf

For a single endomorphism, the characteristic polynomial and the Jordan block decomposition of generalized eigenspaces together reveal all of the information that is independent of coordinates.

The ultimate conclusion of the preceding sections is that the situation for an abstract tableau A . The characteristic sheaf \mathcal{M} knows the dimensions $n, r, (s_1, \dots, s_n)$, as well as all of the dimensions and relationships among the mutual eigenspaces of the various symbol maps. The rank-one cone \mathcal{C} knows the algebraic relationships among the sequences of eigenvalues (which we call ξ), and it also knows on which subspaces the symbol maps commute and on which fail to commute. In summary, \mathcal{M} and \mathcal{C} together know everything about an abstract tableau A that is independent of coordinates.²⁰ Moreover, they are invariant under prolongation!

If the abstract tableau A is a smooth projective bundle, then this applies to involutive Kähler-regular exterior differential systems in the smooth category.

If this formal perspective is appealing, then one might as well dispense with tableaux, symbols, Grassmann bundles, and differential ideals, and instead study the sheaf \mathcal{M} directly, with modern algebraic tools such as [Eis05]. Consider \mathcal{M} as an ideal in $C^\infty(M^{(1)})[u_1, \dots, u_n]$, and consider its free resolution. The Hilbert syzygy theorem

²⁰We revealed this fact using special bases, but that is the same situation as Jordan normal form: there is an abstract structure independent of basis, but it is best revealed by building an adapted basis.

states that there is a finite free resolution that is characterized by its Hilbert polynomial $h_{\mathcal{M}}(d)$. Of course, Theorem 7.11 is reading the leading term of $h_{\mathcal{M}}(d)$!

One might ask how the involutivity of A can be detected as an algebraic property of \mathcal{M} . The answer is tied to Castelnuovo–Mumford regularity, which measures the growth of the Hilbert polynomial. This computation is equivalent to the Cartan characters in Cartan’s test!

While it is not necessarily a useful *computational* tool versus differential forms or tableaux, this perspective allows a broader view of the techniques in PDE analysis, and it suggests that future progress in the field will emphasize on invariant algebraic techniques.

For more on this perspective, see [Mal03], [BCG⁺90, Chapter VIII], and the notes by Mark Green from the 2013 conference *New Directions in Exterior Differential Systems* in Estes Park, Colorado, which are based on the perspective in [CGG09].

Part IV

Eikonal Systems

In Part III, we studied the characteristic scheme defined over $M^{(1)} \subset \text{Gr}_n(TM)$. In this part, we turn our attention to the characteristic scheme as pulled back to an integral manifold $\iota : N \rightarrow M$. This is where the meaning of Ξ as “directions with an ambiguous initial value problem” has clear implications for the internal structure of solutions of a differential equation, as the eikonal system yields intrinsic foliations of integral manifolds N .

12. General Eikonal Systems

First, let us consider the general notion of “eikonal equations” of a projective variety, without specific regard to the characteristic variety.

Consider a smooth manifold N of dimension n . Here are three ways to produce a smooth local hypersurface $H \subset N$:

- The implicit function theorem says that a smooth hypersurface $H \subset N$ is defined locally by a smooth function $f : N \rightarrow \mathbb{R}$, where $T_x H = \ker df$.
- By the Frobenius theorem, this is equivalent to having a local smooth section φ of T^*N such that $d\varphi \equiv 0 \pmod{\varphi}$, for then φ is a rescaling of some df .
- We can also look at the Frobenius theorem from the perspective of Cartan–Kähler theory²¹, as in Theorem 5.3. To make a local function $f : N \rightarrow \mathbb{R}$ or a local section φ of T^*N , consider the jet space $\mathbb{J}^1(N, \mathbb{R})$, which is isomorphic to the bundle $T^*N \times \mathbb{R}$. Jet space is an open neighborhood (or local linearization) of $\text{Gr}_n(N \times \mathbb{R})$ equipped with local coordinates $(x^i, p_i, y) = (x^1, \dots, x^n, p_1, \dots, p_n, y)$ and a contact system \mathcal{J} generated by $\Upsilon = du - p_i dy^i$ and $d\Upsilon$, as in Section 3(a). In these local coordinates, set the independence condition $\omega = dx^1 \wedge \dots \wedge dx^n \neq 0$. Any n -dimensional integral manifold of the exterior differential system $(T^*N \times \mathbb{R}, \mathcal{J}, \omega)$ corresponds to a function $y = f(x^1, \dots, x^n)$ with $p_i = \frac{\partial f}{\partial x^i}$, so we may take $\varphi = df = \frac{\partial f}{\partial x^i} dx^i$. It is easy to see that this exterior differential system has no torsion and has a Kähler-regular tableau with characters $s_1 = s_2 = \dots = s_n = 1$. That is, integral manifolds are parametrized by 1 function of n variables (hardly a surprise).

Now, consider a projective subbundle $\Sigma_N \subset \mathbb{P}T^*N$, meaning it is defined smoothly by homogeneous functions in the local fiber variables (p_i) of T^*N . We want a test that tells us whether there exist hypersurfaces H for which $df \in \Sigma_N$ everywhere. Specifically, we want a theorem like this:

THEOREM 12.1. *Suppose that the eikonal system (defined below) of Σ_N is involutive. Then for any smooth point $\varphi \in \Sigma_{N,x}$, there is a smooth hypersurface $H \subset N$ such that $(T_x H)^\perp = \varphi$ and such that $(T_{\tilde{x}} H)^\perp$ lies in the smooth locus of $\Sigma_{N,\tilde{x}}$ for all $\tilde{x} \in H$.*

Because the hypersurface H and the 1-form φ are not chosen *a priori*, this condition is difficult to interpret using the first two formulations of hypersurfaces; however, the third formulation on $T^*N \times \mathbb{R}$ is well-suited to this theorem. Consider the inclusion $\psi : \Sigma_N \times \mathbb{R} \rightarrow \mathbb{J}^1(N, \mathbb{R})$. The *eikonal system* of Σ is the exterior differential system $\mathcal{E}(\Sigma_N) = \psi^*(\mathcal{J})$ on $\Sigma_N \times \mathbb{R}$; that is, $\mathcal{E}(\Sigma_N)$ is generated by $\psi^*(\Upsilon)$ and $\psi^*(d\Upsilon)$ and has

²¹Although Theorem 5.3 applies as stated only in the analytic category, it can obviously be extended to the smooth category in this case. This sort of extension is explored in Section 14.

independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$. An integral manifold of $\mathcal{E}(\Sigma_N)$ corresponds to a hypersurface in N whose tangent space is annihilated by a section of Σ_N .

We do not prove involutivity of $\mathcal{E}(\Sigma_N)$ in any significant case here; it is typically extremely deep and difficult, and references are provided below. However, the situation in Theorem 12.1 has several interesting consequences and interpretations.

COROLLARY 12.2. *Let $\ell - 1$ denote the projective dimension of Σ_N . In the situation of Theorem 12.1, such hypersurfaces depend on ℓ functions of 1 variable.*

PROOF. Fix $\varphi \in \Sigma_{N,x}$. We work locally (actually, microlocally in the bundle!) near φ , so we may assume N is open, connected, and simply connected, and that $T^*N = N \times \mathbb{R}^n$. Because Σ_N is smooth of affine dimension ℓ in T^*N , we may choose local coordinates (q_1, \dots, q_n) on each fiber of T^*N near such that Σ_N is defined by $q_{\ell+1} = \cdots = q_n = 0$ near φ .

For each $\lambda = 1, \dots, \ell$, let $\sigma^\lambda \in \Sigma_{N,x}$ denote the lines of 1-forms specified as $(0, \dots, 0, q_\lambda, 0, \dots, 0)$, nonzero in the λ slot, in these coordinates. By Theorem 12.1, there is a local hypersurface $H_\lambda \subset N$ and a corresponding local function x^λ such that $dx^\lambda \sim \sigma^\lambda$. Complete x^1, \dots, x^ℓ to a local coordinate system (x^i) on N , and let p_i be the corresponding “derivative” coordinates $p_i = \frac{\partial}{\partial x^i}$ on the fiber of T^*N . Note that $p_i(dx^\ell) = \delta_i^\ell$ by construction, so Σ_N is defined by $p_{\ell+1} = \cdots = p_n = 0$. (Note that the open neighborhood of T^*N around φ may have shrunk during this process.)

Therefore, the contact system on $T^*M \times \mathbb{R}$ is generated in a neighborhood of φ by $\Upsilon = dy - p_i dx^i$, which pulls back to $\Sigma_N \times \mathbb{R}$ as

$$\psi^*(\Sigma_N) = dy - p_\lambda dx^\lambda.$$

The corresponding tableau is the space of $1 \times \ell$ with entries dp_λ for $\lambda = 1, \dots, \ell$, so its Cartan characters are $s_1 = s_2 = \cdots = s_\ell = 1$. \square

This is an interesting proof, using all three perspectives of hypersurfaces. The implicit function theorem on the fiber provides local coordinates on the base by involutivity; then, the Frobenius theorem on the base produces contact coordinates on the fiber that are compatible with the original fiber coordinates. It is easy to adapt this proof to the following corollary, which is useful for constructing coordinates in some situations, as in [Smi14].

COROLLARY 12.3. *For any Σ_N , let $\langle \Sigma_N \rangle$ denote its linear span, which is itself a projective subbundle of $\mathbb{P}T^*N$. If $\mathcal{E}(\Sigma_N)$ is involutive, then $\mathcal{E}(\langle \Sigma_N \rangle)$ is involutive.*

We will now examine several interpretations of the eikonal system that tie together various branches of geometry. Compare Sections 12(a), 12(b), and 13 to [BCG⁺90, V§3(vi)].

12(a). Eikonal Systems as Lagrangian Geometry. The \mathbb{R} term in $T^*N \times \mathbb{R}$ plays little role for the eikonal system $\mathcal{E}(\Sigma_N)$. It is there merely to make obvious the relationship between the eikonal equations and hypersurfaces.

Instead, consider the symplectic manifold T^*N with symplectic 2-form $d\Upsilon$, which is expressed in local coordinates as $d\Upsilon = dp_i \wedge dx^i$ according to Darboux’s theorem.

The *Lagrangian Grassmannian* $LG(N)$ is the bundle over T^*N whose fiber is all the Legendrian n -planes

$$(12.4) \quad LG_\varphi(N) = \{e \in \text{Gr}_n(T_\varphi T^*N) : d\Upsilon|_e = 0\}.$$

Each fiber is isomorphic to the homogeneous space $LG(n, 2n)$, which is the variety of n -planes in $\mathbb{R}[x^1, \dots, x^n, p_1, \dots, p_n]$ on which $dp_i \wedge dx^i = 0$. If we consider a plane $e \in LG(n, 2n)$ for which $dx^1 \wedge \dots \wedge dx^n \neq 0$, then $dp_i = P_{i,j}(e)dx^j$ on e with $P_{i,j} = P_{j,i}$. Hence, the non-vertical open neighborhood of $LG(n, 2n)$ is identified with the space of symmetric matrices, $\text{Sym}^2(\mathbb{R}^n)$.

Suppose the affine subvariety $\Sigma_N \subset T^*N$ is defined smoothly by homogeneous functions in the local fiber variables (p_i) of T^*N . From this perspective, the eikonal system $\mathcal{E}(\Sigma_N)$ is measuring the intersection of $\text{Gr}_n(T_\varphi \Sigma_N)$ with $LG_\varphi(N)$ for all $\varphi \in \Sigma_N$.

COROLLARY 12.5. *The eikonal system $\mathcal{E}(\Sigma_N)$ is involutive if and only if there are local coordinates of T^*N near $\varphi \in \Sigma_N$ in which the non-vertical open set in $\text{Gr}_n(T\Sigma_N) \cap LG(N)$ is described as the $n \times n$ symmetric matrices $P_{i,j}(e)$ that vanish outside the upper-left $\ell \times \ell$ part.*

PROOF. If the eikonal system $\mathcal{E}(\Sigma_N)$ is involutive, then we may construct coordinates as in Corollary 12.2 such that Σ_N is defined by $p_\varrho = 0$ for all $\varrho > \ell$, so $T_\varphi \Sigma_N$ is defined by $dp_\varrho = 0$ for all $\varrho > \ell$. In such coordinates, the open neighborhood of the Lagrangian Grassmannian takes the block form

$$(12.6) \quad \begin{pmatrix} dp_\lambda \\ dp_\varrho \end{pmatrix} \Big|_e = \begin{pmatrix} P_{\lambda,\mu}(e) & P_{\lambda,\varsigma}(e) \\ P_{\varrho,\mu}(e) & P_{\varrho,\varsigma}(e) \end{pmatrix} \begin{pmatrix} dx^\mu \\ dx^\varsigma \end{pmatrix} \Big|_e, \text{ such that } P_{i,j} = P_{j,i},$$

using our index convention (1.8) from Section 1. The condition $e \in T\Sigma_N$ implies $dp_\varrho = 0$, so the lower blocks are zero. The matrix is symmetric, so the upper-right block is zero.

Conversely, suppose such coordinates exist. Then $T\Sigma_N$ satisfies the closed 1-forms $dp_\varrho = 0$, and the dimensions match, so Σ_N satisfies $p_\varrho = \text{constant}$. Since the equations defining Σ_N are homogeneous, it must be $p_\varrho = 0$. Using these coordinates for $T^*N \times \mathbb{R}$ and \mathcal{J} yields $\psi^*(\Upsilon) = dy - p_\lambda dx^\lambda$, as in Corollary 12.2, which is involutive with the correct Cartan characters and gives the desired hypersurfaces in Theorem 12.1. \square

Compare this to Proposition 3.22 in [BCG⁺90, Chapter V]. For more symplectic and Lagrangian geometry, see [Bry93].

12(b). Eikonal Systems as Poisson Brackets. If T^*N describes the state of a physical system, a function $F : T^*N \rightarrow \mathbb{R}$ is called an *observable* [SW86]. The *Poisson bracket* of observables is the operation in local coordinates

$$(12.7) \quad \begin{aligned} \{F, G\} &= \sum_i \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial x^i} \right) \\ &= \sum_i dF \wedge dG \left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial x^i} \right) \end{aligned}$$

The Poisson bracket plays a fundamental in Hamiltonian mechanics and the relationship between symmetries and conservation laws in physics. This is because (12.7) is a Lie bracket on $C^\infty(T^*N)$. (See [Bry93] for details.)

Suppose that O is some set of observables that is closed under linear combinations, and suppose that $\{F, G\} \in O$ for all $F, G \in O$. Then, O is a Lie subalgebra of $C^\infty(T^*N)$ with respect to the Poisson bracket.

Recall that the affine subvariety $\Sigma_N \subset T^*N$ is defined smoothly by observables that take the form of homogeneous functions in the local fiber variables (p_i) of T^*N . For convenience, let us make the additional assumption that the homogeneous functions are algebraic of degree d in (p_i) , so that Σ_N is defined smoothly near $\varphi \in \Sigma_N$ by a set of equations in multi-index form

$$(12.8) \quad 0 = F^\varrho(x, p) = \sum_{|I|=d} f^{\varrho, I}(x) p_I, \text{ for } \varrho = \ell + 1, \dots, n.$$

COROLLARY 12.9. *Let O denote the module in $S = C^\infty(N)[p_1, \dots, p_n]$ generated by (12.8). The eikonal system $\mathcal{E}(\Sigma_N)$ is involutive if and only if $\{O, O\} \subset O$; that is, $\mathcal{E}(\Sigma_N)$ is involutive if and only if the module O is a Lie algebra with respect to the Poisson bracket.*

A proof—which does not depend on the polynomial form (12.8)—can be derived from Corollary 12.5 along with the observation that the Poisson bracket can be defined in a coordinate-free way as the operator such that

$$(12.10) \quad \{F, G\}(\mathrm{d}\Upsilon)^{\wedge n} = n \, \mathrm{d}F \wedge \mathrm{d}G \wedge (\mathrm{d}\Upsilon)^{\wedge(n-1)}.$$

Equations of the form (12.8) appear in analysis as systems of homogeneous first-order PDEs on $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form

$$(12.11) \quad 0 = F^\varrho(x, u, \nabla u) = \sum_{|I|=d} f^{\varrho, I}(x) \frac{\partial u}{\partial x^I}, \text{ for } \varrho = \ell + 1, \dots, n.$$

A famous example is the $n - \ell = 1$ characteristic equation for the wave equation of Section 6(c):

$$(12.12) \quad 0 = -(u_t)^2 + c^2((u_x)^2 + (u_y)^2).$$

This is generalized to any involutive EDS in Section 13.

13. Involutivity of the Characteristic Variety

We would like to apply the entire discussion from Section 12 to the case where Σ_N is a characteristic variety, but first we must establish that Ξ is sensible in T^*N .

Suppose that $\iota : N \rightarrow M$ is a connected integral manifold of an involutive exterior differential system (M, \mathcal{I}) , and that $\iota^{(1)}(N)$ lies in $M^{(1)}$, a smooth and Kähler-regular component of $\mathrm{Var}_n(\mathcal{I})$, as in Section 4.

Fix $x \in N$, and suppose $\iota(x) = p \in M$ and $\iota^{(1)}(x) = e \in M^{(1)}$. For $\xi \in \Xi_e \subset V_e^*$, we can consider the pullback $\iota^{(1)*}(\xi) \in \mathbb{P}T_x^*N \otimes \mathbb{C}$. In a basis (η^i) of T_x^*N , we can write a representative as $\xi = \xi_i \omega^i$ for coefficients $\xi_i \in \mathbb{C}$. As a bundle over N , we have $\iota^{(1)*}(\xi) = \xi_i \eta^i \in \mathbb{P}T^*N \otimes \mathbb{C} = \gamma_N^*$. In this sense, we can pull back the characteristic variety—as a set—to N .

More precisely, recall that Ξ has degree s_ℓ and affine dimension ℓ , but it is a scheme defined by the characteristic sheaf \mathcal{M} . For any local section (u_i) of the coframe bundle $\mathcal{F}_{\gamma^*} \rightarrow M^{(1)}$, we can write the characteristic sheaf \mathcal{M} as a homogeneous ideal in the

module $C^\infty(M^{(1)})[u_1, \dots, u_n]$. At each $e = \iota^{(1)}(x) \subset M^{(1)}$, the affine coframe (u_i) is just a complex basis of e ; therefore, we obtain a basis for $T_x N$ of the form $u_{N,i} = \left(\iota_*^{(1)}\right)^{-1}(u_i)$. To simplify notation, we ignore the subscript N . That is, in some neighborhood of x , the section (u_i) of $\mathcal{F}^* N$ is well-defined. Moreover the stalks of the sheaf $C^\infty(M^{(1)})$ can be pulled back, as $\iota^{(1)*}(f)$ is well-defined for any f defined in a neighborhood of $e = \iota^{(1)}(x)$. Therefore, we can pull back both the coefficients and the coordinates to define the homogeneous ideal \mathcal{M}_N in $C^\infty(N)[u_1, \dots, u_n]$. Let $\Xi_N \subset \mathbb{P}T^*N \otimes \mathbb{C}$ be the scheme defined by \mathcal{M}_N .

Now, the entire discussion from Section 12 applies where Σ_N is any particular component of Ξ_N . We focus our attention on the maximum smooth locus Ξ_N° of Ξ_N . We know additionally that Ξ_N takes the polynomial form (12.8) as derived from (7.14), so it has degree s_ℓ and dimension $\ell - 1$ at smooth points, as a complex projective variety.

THEOREM 13.1 (Guillemin–Quillen–Sternberg). *Suppose that N is an ordinary integral manifold of an involutive exterior differential system \mathcal{I} with Cartan character ℓ . The eikonal system of the smooth locus of the (complex) characteristic variety, $\mathcal{E}(\Xi_N^\circ)$, is involutive. At smooth points in Ξ_N , the characteristic hypersurfaces are parametrized by 1 function of ℓ variables.*

Note that our definition of Ξ_N is the *complex* characteristic variety. This theorem is called the “integrability of characteristics.” Cartan demonstrated several examples of this phenomenon in [Car11]. The proof appears in [GQS70], where a major step is the application of Theorem 7.6. Hence, this result appears to rely in an essential way on all three facets of the characteristic variety seen in Part III.

The converse is *not* true; it is easy to write down non-involutive exterior differential systems for which $\mathcal{E}(\Xi_N)$ is involutive.

However, in [Gab81], Ofer Gabber proved a more general form of Theorem 13.1 that was conjectured in [GQS70] that removes practically all of the technical assumptions and recalls the ideas of Section 12(b).

THEOREM 13.2 (Gabber). *Let S be a filtered ring whose graded ring $gr(S)$ is a Noetherian commutative algebra over \mathbb{Q} . Let M be a $gr(S)$ -ideal that is finitely generated as an S -module. Then $\{\sqrt{M}, \sqrt{M}\} \subset \sqrt{M}$*

In our context, Gabber’s theorem applies to the case where $S = C^\infty(N)[p_1, \dots, p_n]$, the ring of polynomials in local fiber variables of T^*N , filtered by degree. Then, $gr(S)$ is the ring of homogeneous polynomials, graded by degree, which admits a Poisson structure like (12.7). The $gr(S)$ -ideal M is the characteristic sheaf \mathcal{M}_N , which by (7.14) is defined by homogeneous polynomials if the original exterior differential system is involutive. By Hilbert’s Nullstellensatz, the radical ideal \sqrt{M} defines the generic component Ξ_N° . Thus, the conclusion $\{\sqrt{M}, \sqrt{M}\} \subset \sqrt{M}$ invokes Corollary 12.9 to say that the eikonal system $\mathcal{E}(\Xi_N^\circ)$ is involutive.

From the general discussion of eikonal systems surrounding Theorem 12.1, the interpretation of these theorems is apparent:

COROLLARY 13.3. *Suppose that N is an ordinary integral manifold of an involutive exterior differential system \mathcal{I} with Cartan character ℓ . Then N admits a local—possibly complex—coordinate system (x^1, \dots, x^n) such that $dx^1, \dots, dx^\ell \in \Xi_N$.*

In [Smi14], the linear span of the characteristic variety, $\langle \Xi_N \rangle$ is studied in comparison to the Cauchy retraction space, \mathfrak{g}_N^\perp . Suppose that the affine dimension of $\langle \Xi_N \rangle$ is L and that the affine dimension \mathfrak{g}_N^\perp is ν . These spaces are nested, so $\ell \leq L \leq \nu \leq n$.

COROLLARY 13.4. *Suppose that N is an ordinary integral manifold of an involutive exterior differential system \mathcal{I} with Cartan character ℓ . Then N admits a local—possibly complex—coordinate system (x^1, \dots, x^n) such that $dx^1, \dots, dx^\ell \in \Xi_N$, such that $dx^{\ell+1}, \dots, dx^L \in \langle \Xi_N \rangle$, and such that $dx^{L+1}, \dots, dx^\nu \in \mathfrak{g}_N^\perp$.*

Corollary 13.4 is a simple result, but its proof relies on building a coframe of N in which the nilpotent parts of the commuting symbol maps B_I^λ are identified clearly; that is, it depends in an essential way on Theorems 13.1 and 5.4. The key point is that it reinforces the following dogma:

REMARK 13.5 (General Dogma of the Characteristic Variety). An exterior differential system (M, \mathcal{I}) is a geometric object over M , meaning that its key properties are coordinate-invariant. On each Kähler-regular component $M^{(1)}$, knowing this geometry is equivalent to knowing the characteristic scheme and rank-one variety over $M^{(1)}$, which are prolongation-invariant. Moreover, the geometry of an EDS imposes a geometry on its solutions, $\iota : N \rightarrow M$, and this imposition is also dictated by the characteristic scheme and rank-one variety. Therefore, exterior differential systems can be classified up to coordinate equivalence as “parametrized families of manifolds N with associated characteristic geometry.”

This is not a theorem; it is an attitude.

To make Remark 13.5 robust for a general exterior differential system, the scheme separating $\text{Var}_n(\mathcal{I})$ into its components $M^{(1)}$ —each component smooth with its own fixed Cartan characters over some subvariety of M —would have to be studied, and very little progress has been made at that level of abstraction. None-the-less, whenever some property of PDEs is encountered, Remark 13.5 urges us to ask “is this property really invariant, or an artifact of my coordinates?” which is best answered by asking “can this property be reinterpreted using the characteristic scheme?” Sections 14 and 15 discuss progress of this type.

14. Yang’s Hyperbolicity Criterion

One of the great frustrations of the Cartan–Kähler theorem is that it relies on the Cauchy–Kowalevski theorem, so it applies only in the analytic category. One can see its dramatic failure in the smooth category in [Lew57]. However, this frustration has been escaped in some special cases by exploiting the structure²² of Ξ . For example:

- **ODE Systems.** Suppose that (M, \mathcal{I}) is involutive over C^∞ and that $\Xi = \emptyset$. Then $\ell = 0$, so the tableau A is the trivial (irrelevant) subspace of $W \otimes V^*$. The prolonged system $\mathcal{I}^{(1)}$ on $M^{(1)}$ is Frobenius, and $M^{(1)}$ is merely a copy of M whose fiber is the unique element of an integrable distribution. That integrable distribution is the Cauchy retraction space \mathfrak{g} of \mathcal{I} as in Section 5(b),

²²If we take the broadest possible interpretation of Remark 13.5 to heart, then *any* possible escape from analyticity ought to arise from the structure of Ξ . However, the reader is cautioned again that a dogma is not a theorem.

so it must have been that $\mathcal{I} = \mathfrak{g}^\perp$. The flow-box theorem foliates M by solutions in the smooth category. (Actually, in the Lipschitz category, by standard ODE theory!) If N is a leaf of this foliation, then removing Cauchy retractions on the original exterior differential system (M, \mathcal{I}) yields the exterior differential system $(N, 0)$.

- **Empty Systems.** Suppose that (M, \mathcal{I}) is involutive over C^∞ and that $\Xi = V^*$ with $(s_1, s_2, \dots, s_n) = (r, r, \dots, r)$. Then, the tableau A is the total space $W \otimes V^*$. Therefore, $M^{(1)}$ is an open domain in $\text{Gr}_n(TM)$, so $\mathcal{I} = 0$, and there is no condition whatsoever²³ on integral manifolds $\iota : N \rightarrow M$; however, the prolongation $\iota^{(1)} : N \rightarrow M^{(1)}$ would have to satisfy the contact ideal, forcing some regularity on N . We studied this EDS in Section 2.

A less trivial special case is presented in [Yan87], which is the subject of this section.²⁴

A tableau A is called *determined* if $s_1 = s_2 = \dots = s_{n-1} = r$ and $s_n = 0$; that is, $s = (n-1)r$, so $t = r$, and $H^1(A) \cong W$. Cartan's test shows that a determined tableau is always involutive, so we may assume that A is written in endovolutive form as in Theorem 5.4. The only nontrivial symbol endomorphisms in (1.20) are $B_\lambda^\lambda = I_{r \times r}$ and B_n^λ for $\lambda = 1, \dots, n-1$, like this:

$$(14.1) \quad (B_i^\lambda) = \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 & B_n^1 \\ & I_r & 0 & \cdots & 0 & B_n^2 \\ & & \ddots & & \vdots & \vdots \\ & & & I_r & 0 & B_n^{n-2} \\ & & & & I_r & B_n^{n-1} \end{bmatrix}.$$

The quadratic involutivity condition is trivial, which is why Cartan's test passes automatically.

LEMMA 14.2. *Suppose A is determined and written in endovolutive bases. Identify $H^1(A)$ with W , and use our endovolutive basis of W for both. Then for any $\varphi \in V^*$, the symbol map $\sigma_\varphi : w \mapsto \sigma(w \otimes \varphi)$ from Section 6(b) is*

$$(14.3) \quad \sigma_\varphi = \left(\varphi_\lambda B_n^\lambda - \varphi_n I \right).$$

Then

$$(14.4) \quad \ker \sigma_\varphi = \ker \left(\varphi_\lambda B_n^\lambda - \varphi_n I \right),$$

and the characteristic ideal \mathcal{M} is generated by

$$(14.5) \quad \det \sigma_\varphi = \det \left(\varphi_\lambda B_n^\lambda - \varphi_n I \right).$$

In particular, $\xi \in \Xi$ if and only if ξ_n is an eigenvalue of $\xi_\lambda B_n^\lambda$.

²³The most extreme and amusing exploitations of the flexibility of $\text{Gr}_n(TM)$ come from the homotopy principle [Gro86, EM02].

²⁴As it happens, the attempt to understand [Yan87] in the context of [BCG⁺90, Chapter VIII] was the inspiration for computing the details shown in [Smi15] and the entire approach of these notes.

PROOF. The first two equations are immediate from our block form. From Part III, we know $w \otimes \xi \in A$ if and only if $B(\xi)(v)w = \xi(v)w$ for all v . Therefore, we compute in our endovolutionary basis

$$\begin{aligned}
 \xi(v)w &= B(\xi)(v)w \\
 &= \xi_\lambda v^i B_i^\lambda(w) \\
 (14.6) \quad &= (\xi_\lambda v^\lambda)w + \xi_\lambda v^n B_n^\lambda w \\
 &= (\xi(v) - \xi_n v^n)w + \xi_\lambda v^n B_n^\lambda w
 \end{aligned}$$

That is, $\xi_n w = \xi_\lambda B_n^\lambda w$. □

COROLLARY 14.7. Consider a determined tableau as in Lemma 14.2. Fix an integral element e . Suppose that e' is a real hyperplane in e such that $(e')^\perp \otimes \mathbb{C} = \varphi \in V^*$ and $\varphi \notin \Xi$. Then $\sigma_\varphi : W \rightarrow H^1(A)$ is an isomorphism.

PROOF. By Lemma 14.2, we have $\ker \sigma_\varphi \neq 0$ if and only if $\varphi \in \Xi$. □

DEFINITION 14.8. Suppose e' is a real hyperplane in e corresponding to the real covector $\varphi = (e')^\perp \in \mathbb{P}e^*$. The real hyperplane is called *space-like* if:

- (i) $\varphi \otimes \mathbb{C} \notin \Xi_e$, and
- (ii) For any $\eta \in \mathbb{P}e^*$, there is a real basis of W in which $(\sigma_\varphi)^{-1}(\sigma_\eta) : W \rightarrow W$ is real and diagonal, and
- (iii) that choice of basis is a smooth function of $[\eta] \in e^*/\varphi = (e')^*$.

A determined symbol A is called *determined hyperbolic* if V admits a (real) space-like hyperplane.

Here is a simple example using our notation from Lemma 14.2: Fix $n = 3$. To meet the first condition, suppose that $\varphi = 1u^1 + 0u^2 + 0u^3$ is not in Ξ . Then $\sigma_\varphi = B_3^1$, and 0 is not an eigenvalue of σ_φ , which of course implies $\sigma_\varphi = B_3^1$ is invertible. Say $\eta = 0u^1 + 1u^2 + \tau u^3$, so that $\sigma_\eta = B_3^2 - \tau I_r$. The second condition is that $(B_3^1)^{-1}(B_3^2 - \tau I_r)$ is diagonalizable using some basis-change g_τ . The third condition is that g_τ is continuous in the projective variable τ . Suppose moreover that we take our basis such that the basis-change at $\tau = 0$ is $g_0 = I$. Then we have the condition that $(B_3^1)^{-1}B_3^2$ is a diagonal matrix, D . This puts restrictions on the possible forms of these matrices. For example, $\ker B_3^2 = \ker D$ and $\text{im } B_3^2 \subset \text{im } B_3^1$.

DEFINITION 14.9. A tableau is called *hyperbolic* if V admits a flag given by a basis (u^1, \dots, u^n) of V^* such that the each of the sequential initial value problem from $\langle u^i, \dots, u^n \rangle^\perp$ to $\langle u^{i+1}, \dots, u^n \rangle^\perp$ has a hyperbolic determined tableau.

THEOREM 14.10 (Yang). Theorem 5.3 applies in the smooth category, if A is hyperbolic.

The proof proceeds by replacing the Cauchy–Kowalevski initial-value problem with the Cauchy initial-value problem for determined first-order quasilinear hyperbolic PDEs. See [Yan87] and Appendix A of [Kam89] for more details.

Clearly the definition of *hyperbolic* depends on the geometry of Ξ and the symbol maps B_i^λ ; however, to my knowledge no one has succeeded in writing down the explicit

criteria on B_i^λ or \mathcal{C} or Ξ for general hyperbolicity. Hence, Yang’s condition is not yet available to computer algebra systems. If that can be accomplished, it means we can identify a subvariety of the moduli of involutive tableaux—as in Section 5(a)—that admit solutions in the smooth category.

One well-understood special case is when $\ell = 1$, so Ξ_e contains s_1 real points (with multiplicity). If the number of distinct points is sufficiently large (greater than n), then this is the situation for hyperbolic systems of conservation laws, as in [Tsa91]. The eikonal system is rigid, so each solution is foliated by s_1 characteristic hypersurfaces. Multiplicity corresponds to nilpotent pieces of the generalized eigenspaces of the symbol endomorphisms B_i^1 . See again Section 8.

15. Open Problems and Future Directions

Our perspective here has been simple-minded—focussing on matrices and their computable properties—to gain intuition of Ξ and $\mathcal{C}(\Xi)$ as rapidly as possible. The articles [Smi14] and [Smi14] are founded on this perspective, but reveal additional detail in the structures discussed here. For more modern and sophisticated treatment, please see [Mal03], [KL07], and [CGG09]. Additionally, Chapters V–VIII of [BCG⁺90] contain significantly more results than we have summarized here.

To conclude, here are some interesting questions which—to my present knowledge—are open subjects. I believe that they represent the major theoretical gaps in the subject of exterior differential systems. They are worth serious consideration as research projects, and offer great opportunities for collaboration between analysts, differential geometers, algebraic geometers, and scientific programmers.

- (i) Variety of Involutive Tableaux. For given r , n , and Cartan characters (s_1, \dots, s_ℓ) , what is the variety of involutive tableaux (with fixed coefficients)? Can we compute its dimension or degree or Hilbert polynomial? Section 5(a) demonstrates a first step toward understanding the variety of involutive tableaux, as Theorem 5.4 gives the ideal in certain bases. However, to answer the question completely, one would need to examine how the coefficients in (5.6) change under arbitrary basis changes of V^* and W .
- (ii) Special Hyperbolic Integrability Criteria. Solution techniques (such as Lax pairs, inverse scattering, hydrodynamic reduction, and Bäcklund transformations) play a key role in the analysis of wave-like PDEs, especially those coming from physics and geometry. Given that these techniques are coordinate-invariant, the Dogma 13.5 suggests that they should all be expressible as algebraic conditions on \mathcal{M} . Expressing those conditions in an abstract way over Ξ and $M^{(1)}$ would allow more systematic geometric approach to many of the *ad-hoc* methods in the analysis of PDEs.²⁵
- (iii) Elliptic Systems. Consider the classical results regarding elliptic regularity of quasilinear elliptic operators. This is another form of “special integrability

²⁵Indeed, the central theme of the conference for which these notes were prepared was to express Ferapontov’s notion of hydrodynamic integrability in terms of algebro-geometric structures in the Lagrangian Grassmannian. The notion of hydrodynamic integrability is tied completely to the secant variety of \mathcal{C} .

criteria.” How far can the notion of elliptic regularity be extended to general exterior differential systems? Certainly the conditions of involutivity, $\langle \Xi \rangle = V^*$, and $\Xi_{\mathbb{R}} = \emptyset$ are necessary, and one can directly translate the classical theorems to an EDS written specifically to describe a quasilinear second-order elliptic operator in local coordinates, but what other technical assumptions can be dropped? Some discussion appears in [BCG⁺90, Chapter X§3].

- (iv) **Moduli of Involutive Tableaux.** Refining the first problem in light of the second and third problems, can we identify invariant sub-varieties of the variety of involutive tableaux? Dogma 13.5 indicates that we should be able to identify subvarieties, such as hyperbolic tableaux, elliptic tableaux, systems satisfying special integrability conditions, and so on. What does it mean when these sub-varieties intersect? What Where do the Lewy example fall in this variety? Are there other sub-varieties that have not been observed in classical equations? If there is any organizing geometry behind the “nearly impenetrable jungle” of involutive PDEs, this is where we should look.
- (v) **Weakness of Involutivity of Characteristics.** Note that Theorem 13.2 does not regard the involutivity of an exterior differential system in any direct way; the assumption of involutivity of \mathcal{I} enters Theorem 13.2 only because we know that \mathcal{M}_N is an ideal of homogeneous polynomials from (7.14). Thus, we expect that the condition “ Ξ_N is the characteristic scheme of an exterior differential system \mathcal{I} , and $\mathcal{E}(\Xi_N)$ is involutive” is much weaker than “ Ξ_N is the characteristic scheme of an exterior differential system \mathcal{I} , and \mathcal{I} is involutive.” The gap between these two statements is extremely important to explore, as it goes to the heart of the question about how involutivity leads to solutions of the initial-value problem for a system of PDEs. To put this a different way, can we construct an embedded variety $\Xi_N \subset \mathbb{P}T^*N$ that is involutive, but for which there is *no* involutive exterior differential system for which Ξ is the characteristic variety?
- (vi) **Global Integrability of the Characteristic Variety.** If A is involutive, then the system $\mathcal{E}(\Xi_N^o)$ is involutive on an ordinary integral manifold, N . However, it is not clear whether there is sense in which Ξ^o is involutive over $M^{(1)}$ itself. That is, consider the EDS on $M^{(1)}$ generated by adding a smooth section ξ to $\mathcal{I}^{(1)}$. Under what circumstances is this involutive? Can Gabber’s theorem be adopted to this case? This has theoretical implications for special integrability conditions (below). Additionally, given its algebraic nature, can Gabber’s theorem provide solutions for certain types of PDEs with low regularity?
- (vii) **The Prolongation Theorem.** Does prolongation always uncover solutions of an exterior differential system, if we remove the regularity assumptions on $M^{(1)}$ and consider the many components of the scheme $\text{Var}_n(\text{Var}_n(\cdots(\mathcal{I})\cdots))$? As experts are well aware, this has been the key open question in the subject for most of a century. (See [BCG⁺90, Chapter VI].) In the context of this monograph, the question is related to whether the block-form of involutive tableau (1.20) and the involutivity conditions of Theorem 5.4 can be extended from $M^{(1)}$ to non-smooth points in $\text{Var}_n(\mathcal{I})$? Because of the interaction of Guillemin normal form and involutivity with Spencer cohomology as in Section 9,

such an extension of the endovolutive block-form could be helpful in an effort to construct (or prove the non-existence of) counterexamples.

- (viii) Representation Theory of Lie pseudogroups. Lie pseudogroups are subgroups of the diffeomorphism pseudogroup whose trajectories are the solutions of involutive PDEs. See [Olv09]. Just as Jordan form (in the guise of the Levi decomposition) is the key first step toward understanding the representation of Lie groups, it is reasonable to expect that the endovolutive block-form (1.20) and Theorem 5.4 can serve as the foundation of a representation theory of Lie pseudo-groups. Any results regarding the “moduli of involutive tableaux” can be applied to Lie pseudogroups with those tableaux. Indeed, the first application of Theorem 13.1 was the classification of the primitive Lie pseudogroups [QSS66].

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